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Gabor’s signal expansion in optics

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ABSTRACT – In this chapter some applications of Gabor’s signal expansion in the field of optics are considered. After a preparatory treatment of some necessary optics fundamentals and the translation of relevant concepts of time-dependent signals to signals that depend on spatial variables, Gabor’s signal expansion and its companion – the Gabor transform – are introduced in the field of optics. Special attention is paid to Gaussian windows, which are related to the well-known concept of Gaussian light beams in optics. The case of critical sampling is considered, in particular in its relation to the degrees of freedom of an optical signal and to the space-bandwidth product of an optical system; to do this, the propagation of Gabor’s expansion coefficients through optical systems is considered. The case of integer oversampling is considered and it is shown how in that case the Gabor transform can be transformed into a product of Zak transforms. It is demonstrated how this product of Zak transforms can form the basis of a coherent-optical setup for generation of the Gabor transform and it is shown how this setup can be used for an approximate generation of the windowed Fourier transform.

14.1 Introduction

In his original paper, Gabor suggested the representation of a time signal in a combined time-frequency domain. Actually he proposed to represent the signal as a superposition of shifted and modulated versions of a so-called elementary signal. Moreover, as an elementary signal he chose a Gaussian signal, because such a signal has a good localization, both in the time domain and in the frequency domain.

In this chapter we will consider some applications of Gabor’s ideas in the field of optics. In optics, signals not only depend on the time variable \( t \), but also on the space vector \( \mathbf{r} = (x,y,z) \). In fact, the space dependence is often much more important than the time dependence. To see how Gabor’s ideas can be translated to the spatial domain, we shall confine ourselves in this chapter to strictly time-harmonic optical signals with temporal frequency \( \omega \). Such an optical signal can be described, for instance,
by \( \mathbb{R}\{\varphi(x, y, z) \exp(-j \omega t)\} \), where \( \mathbb{R} \) denotes the real part and where the complex amplitude \( \varphi(x, y, z) \) contains the relevant spatial information of the signal.

Very often the optical signal propagates through some optical medium from a certain input plane \( z = z_i \), say, to an output plane \( z = z_o \). In that case it suffices to know – in an arbitrary plane \( z = \text{constant} \) – the complex amplitude \( \varphi(x, y) \) that depends on the transverse coordinates \( x \) and \( y \) only; we will see in Section 14.2 that the \( z \)-dependence of the complete signal \( \varphi(x, y, z) \) follows from the properties of the medium in which the optical signal is propagating. In Section 14.2 we will also show how we can interpret the two-dimensional Fourier transform \( \hat{\varphi}(u, v) \) of a function \( \varphi(x, y) \) in physical terms. Throughout this entire chapter we will denote the Fourier transform of a function by the same symbol as the function itself, but marked by a hat on top of the symbol.

In Section 14.3 we will consider Gabor's signal expansion and its inverse – the Gabor transform, with the help of which Gabor's expansion coefficients can be determined – for optical signals. For convenience we will consider optical signals that depend on one transverse coordinate \( x \) only, and that do not depend on \( y \). In that case we can restrict ourselves to the one-dimensional function \( \varphi(x) \) and its Fourier transform \( \hat{\varphi}(u) \). The extension to the more general, two-dimensional case is rather straightforward. We will pay special attention to the case of a Gaussian elementary signal, which is intimately related to the optically important Gaussian beam. Moreover, we will use Section 14.3 to do some preparatory, theoretical work on critical sampling, on integer oversampling and the product form of the Gabor transform in terms of the Zak transform, and on the windowed Fourier transform, expressed as an interpolation of the Gabor transform; the results of this work will then be used in Sections 14.4 and 14.5.

The propagation of an optical signal, described in terms of its Gabor coefficients, will be treated in Section 14.4. In this section we will restrict ourselves to the case of critical sampling of the space-frequency domain, and we will study in more detail the concept of degrees of freedom of a signal.

Finally, in Section 14.5 a coherent-optical setup will be considered, with which Gabor's expansion coefficients can be generated.

### 14.2 Some optics fundamentals

In this section we will show how the exponential \( \exp[j(k_x x + k_y y)] \) that plays a central role in the Fourier transform and in Gabor's expansion of an optical signal, can be given a physical interpretation. We will derive such an interpretation considering only one of the simplest systems in which an