Aspects of Gabor analysis on locally compact abelian groups

Karlheinz Gröchenig

ABSTRACT – Motivated by recent formulations of Gabor theory for periodic and for discrete signals, this chapter develops several aspects of Gabor theory on locally compact groups. First an uncertainty principle in terms of the short time Fourier transform is derived (Lieb’s inequalities). It captures the intuition that any signal occupies a region in the time-frequency plane of area at least one. Secondly, the Zak transform, introduced on locally compact abelian groups already by A. Weil, is used to analyze Gabor frames in the case of integer-oversampled lattices in the time-frequency plane. In this context it is observed that the Balian–Low theorem depends on the group structure and that the known versions do not hold for discrete and compact groups. In the final section a notion for the density of lattices in defined and necessary conditions for lattices in the time-frequency plane to generate Gabor frames are derived.

6.1 Introduction

Gabor theory tries to express and understand the time-frequency behavior of functions in terms of time-frequency shifts $e^{2\pi i u}g(t-x)$ of a single (or several) function(s) $g$. According to the prevailing fashion $g$ is called a window, Gabor atom, Gabor logon, time-frequency template, Weyl-Heisenberg atom, Gabor wavelet among others.

Usually Gabor theory is investigated on $\mathbb{R}$, but recently other settings have been looked at. Working with discrete signals, Gabor theory is done on $\mathbb{Z}$ [Li94a, Jan94a, ML94, ZZ93a], and numerical implementations require to consider finite periodic signals and consequently Gabor theory on finite cyclic groups [FCS95, Orr93a, Orr93b, Pri96b, QC93, RN96, WR90]. In image processing Gabor theory on $\mathbb{R}^2$ or $\mathbb{R}^d$ and in its discrete versions on $\mathbb{Z}^d$ and finite abelian groups is necessary [AZG91, Dau88b, Li94c]. A computer scientist might even argue that the right setting for Gabor theory are the $p-\text{adic}$ groups, because their group laws imitate the computer arithmetic most closely. Since Gabor theory rests mainly on the structure of translations and modulations, it is possible to extend it to other abelian groups.
The theory and the formal computations on all these groups are always the same, and their derivation becomes somewhat repetitive, often with unnecessary notational problems. In a sense, present-day Gabor theory resembles the state of abelian harmonic analysis in the 1930’s before it was discovered that all but a few theorems on Fourier integrals and Fourier series could be formulated for general locally compact abelian (LCA) groups.

One of the objectives of this chapter is to present some of the main results of Gabor theory for LCA groups. Usually Gabor theory on \( \mathbb{R} \) is more difficult than on discrete or compact groups, because technical questions about convergence of series and integrals and boundedness of certain operators do not occur or are much easier to deal with. The generalization of Gabor theory from \( \mathbb{R}^d \) to LCA groups is then routine and can be based on standard harmonic analysis on LCA groups [HR63, Rei68]. The results obtained here are “new”, but hardly surprising.

Our second objective are the occurring differences due to the group structure. On \( \mathbb{R} \) the uncertainty principle and the Balian-Low theorem (“BLT”) are fundamental obstructions to ideal time-frequency localization and much of the research in Gabor theory is driven by the need to live with these obstacles. However, on general LCA groups neither the uncertainty principle nor the BLT hold true in their standard formulations. It is still a challenge to find appropriate formulations of these principles that validate our fundamental intuition about time-frequency concentration on other groups than \( \mathbb{R}^d \). This leads to new interesting problems, and the transition from the discrete case (necessary for the numerical analysis) to the continuous case (the “real world”) with its subtleties seems to need rethinking and many aspects have yet to be fully understood.

Section 2 presents the required facts about harmonic analysis on LCA groups and serves as a warm-up for Gabor theory on LCA groups.

In Section 3 we shall discuss Lieb’s inequalities [Lie90] for the short time Fourier transform. They imply that the time-frequency concentration of a signal in the time-frequency plane has an area of size at least one, and thus can be viewed as an alternative formulation of the uncertainty principle. Since the proof uses only properties of the short time Fourier transform, the inequalities of Young and Hausdorff-Young, these inequalities carry over to general LCA groups. Furthermore, on \( \mathbb{R}^d \) the minimizing functions are Gaussians as in the classical uncertainty principle. Thus the minimizing functions on general groups could serve as a good substitute for the Gaussian.

Section 4 treats the theory of critically sampled and integer oversampled Gabor frames. As the main tool we use the Zak transform. Although the Zak transform is a widely applied tool in signal analysis, its mathematical origin in A. Weil’s work [Wei64] in 1964 seems to be largely forgotten. Currently one can observe the rediscovery of the Zak transform on groups such as \( \mathbb{Z} \) or finite cyclic groups, although Weil had already investigated