CHAPTER VII

Formations in Space

Vectors

49. In space we fix at random a volume $\Omega$, to which we will give the name the unit volume; we will indicate by $\omega$ the trivector of this volume, that is that trivector such that if $O$ is any point in space, one has $O\omega = \Omega$, and we will call it the unit trivector.

If $A$ is a formation of the first species, $A = m_1A_1 + \ldots + m_nA_n$, in which $A_1 \ldots A_n$ are points, one deduces

$$\frac{A\omega}{\Omega} = m_1 + \ldots + m_n,$$

that is $\frac{A\omega}{\Omega}$ represents the mass of the system $A$. The equation $A\omega = 0$ says that $A$ is a vector. If $A\omega$ is not zero, $\frac{\Omega}{A\omega} A$ represents the simple point coincident with $A$. If $A$ is a volume, $\frac{A}{A\omega}$ represents the trivector of the volume.

50. We will begin by concerning ourselves with vectors and their products, that is bivectors and trivectors. If it is demonstrated that $IJK$ are three vectors, such however that $IJK$ is not zero, every other vector $U$ can be cast into the form

$$U = xI + yJ + zK.$$

The numbers $x, y, z$ are called the coordinates of the vector $U$ with respect to the three reference vectors $I, J, K$.

Multiplying the preceding equation by $JK, KI, JK$, one recovers

$$x = \frac{UJK}{IJK}, \quad y = \frac{UKI}{IJK}, \quad z = \frac{UIJ}{IJK},$$

which express the coordinates of the vector $U$ as ratios of trivectors.

Suppose one substitutes the values of $xyz$ given by (2) into (1); one has, after some reductions,

$$\frac{IJK}{\omega} U = \frac{UJK}{\omega} I + \frac{UKI}{\omega} J + \frac{UIJ}{\omega} K.$$

To simplify the expression we agree to suppress the unit trivector in the denominator, understanding, from now on, that $IJK$ is no longer the trivector product of $I, J, K$, but the ratio of this trivector to the unit trivector.

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Now one can write the preceding formula as

\[ IJK.U = UJK.I + UKI.J + UIJ.K, \]

a relation between any four vectors.

If \( U = xI + yJ + zK, \ V = x'I + y'J + z'K, \) one recovers for the bivector \( UV \) the expression

\[ UV = (yz' - y'z)JK + (zx' - z'x)KI + (xy' - x'y)IJ. \]

One sees from this that every bivector \( u \) can be cast into the form

\[ u = uJK + vKI + wIJ. \]

The numbers \( u, v, w \) are called the coordinates of the bivector \( u \); formula (4) gives the coordinates of the bivector \( UV \) as functions of the coordinates of its factors.

If the vector \( U \) and the bivector \( u \) are given by (1) and (5) one recovers

\[ uu = (xu + yv + zw)JK. \]

If \( x, y, z, x', y', z' \) are the coordinates of the three vectors \( U, V, W, \) one recovers

\[ UVW = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} IJK. \]

51. If, as in N. 44, to the expression \( UV.U'V' \) is attributed the meaning

\[ UV.U'V' = UU'V'.V - VUV'.U, \]

one has as well, by virtue of identity (3) of the preceding,

\[ UV.U'V' = UVV'.U' - UVU'.V', \]

whence \( UV.U'V' \) represents a vector that lies on the two bivectors \( UV \) and \( U'V' \); we can call it the regressive product, or intersection, of the two bivectors. If the bivectors are represented by \( u \) and \( v \), their regressive product can be represented as \( u.v \) or \( uv \).

For the regressive product of bivectors there exist all the identities that hold for regressive products of lines and planes, that is to say:

\[ UV.u = Uu.V - Vu.U. \]
\[ u.v = -v.u. \]
\[ (u = u') \cap (v = v') \leq (u.v = u'.v'). \]
\[ (u + u').v = u.v + u'.v; \quad u.(v + v') = u.v + u.v'. \]
\[ (ku)v = u(kv) = k(uv). \]
\[ UV.UW = UVW.U. \]