Panorama of grid-bound self-affine variability

♦ Abstract. Multifractals and 1/f noises arose independently, as will be seen in Chapter N2. Therefore, it is not surprising that they continue near-unanimously to be viewed as separate and unrelated concepts.

This chapter argues the opposite case: that both concepts are specific aspects of a far broader underlying reality denoted as "wild self-affine variability." The term "self-affine" may be unfamiliar to some readers, but everyone soon perceives it as being self-explanatory. It was first used in M1977F, in an entry reproduced with little change in M1982F[FGN].

To achieve its goal, this chapter describes straightforward constructions that are recursive, therefore automatically yield self-affine outputs. They are "grid-bound," that is, constrained to a recursively refined grid, in the style of the devil staircase, the binomial measure and other familiar proto-fractal constructions. Hence, they are sharply simplified and will be referred to as "cartoons" of reality.

Nevertheless, their outputs include the following structures:

A ) Cartoons of the (ordinary) Wiener Brownian motion and of its fractional Brownian generalizations investigated in M1998H.

B ) Cartoons of varied forms of 1/f noise, some of them already familiar and others less so or not at all.

C ) The simplest multifractals, namely, C') the binomial and the multinomial measures and the "microcanonical" random measures investigated in M1974f[N15] as well as C") the cartoons of oscillatory variation investigated in M1997E.

Much of this book deals with multifractals and 1/f noises that are immensely more realistic than any cartoon, because they are gridless. One example is the original limit lognormal multifractal introduced in
However, those gridless constructions are not simply related to one another. The cartoons include the essentials of the topics treated in this book and in its companions (M1997E, M1998H, and M1998L). The word Panorama in the title is justified more fully (perhaps to excess!) because this chapter glances towards further forms of self-affine variation that have not been "visited" yet. The cartoons underline both the similarities and the differences between those various cases.

For self-similar shapes, the key parameter is fractal dimension, a concept that keeps splitting into ever-new variants. For self-affine shapes, the same role is played by an exponent $H$. It has modest double origins in the work of Hölder and Hurst. But it has taken off on its own and branched out, and now keeps appearing under an increasingly wide range of distinct roles. Section 1 describes many examples as introductions to a general definition of the diagonal-axial self-similar cartoons. For most of those examples, $H$ has a single implementation.

Section 2 takes up the general idea behind $H$ from a very different angle and introduces important new variants and the corresponding terminology. The most general diagonal-axial cartoons will be called multifractal. Those in an important subclass will be called unifractal. A different important subclass will be called mesofractal.

Section 3 is perhaps the most novel in this chapter. It throws light on the nature of cartoons, therefore of multifractals and $1/f$ noise, by transforming clock time in diverse ways. Section 3.2 introduces a trail exponent $H_T$, in effect, by projecting on a space orthogonal to the time axis: this amounts to adopting as time any monotone increasing function of clock time. Section 3.3 finds it convenient to play with "isochrone time." Section 3.4 introduces the notion of "intrinsic time." Section 3.5 describes a simple but fundamental result dubbed "baby theorem." It permits every cartoon to be represented as a compound function: a unifractal cartoon of a multifractal intrinsic time.

Section 4 describes graph and spectrum implementations of $H$.

Section 5 comments briefly on several distinct topics.

In the landscape of mathematics and physics, multifractals are often perceived as an isolated island ruled by an esoteric formalism. I believe, to the contrary, that it is useful to think of them as being special examples of a far broader natural and mathematical phenomenon that occurs in diverse fields. Different instances having been identified independently, they go under a variety of distinct names. Physicists and engi-