Chapter 6

Stability and weak stability

The stability problem is the question of to what extent the conclusion of a theorem is sensitive to small changes in the assumptions. Such description is, of course, vague until the questions of how to quantify the departures both from the conclusion and from the assumption are answered. The latter is to some extent arbitrary; in the characterization context, typically, stability reasoning depends on the ability to prove that small changes (measured with respect to some measure of smallness) in assumptions of a given characterization theorem result in small departures (measured with respect to one of the distances of distributions) from the normal distribution.

Below we present only one stability result; more about stability of characterizations can be found in [73, Chapter 9], see also [102]. In Section 6.2 we also give two results that establish what one may call weak stability. Namely, we establish that moderate changes in assumptions still preserve some properties of the normal distribution. Theorem 6.2.2 below is the only result of this chapter used later on.

6.1 Coefficients of dependence

In this section we introduce a class of measures of departure from independence, which we shall call coefficients of dependence. There is no natural measure of dependence between random variables; those defined below have been used to define strong mixing conditions in limit theorems; for the latter the reader is referred to [65]; see also [10, Chapter 4].

To make the definition look less arbitrary, at first we consider an infinite parametric family of measures of dependence. For a pair of \( \sigma \)-fields \( \mathcal{F}, \mathcal{G} \) let

\[
\alpha_{r,s}(\mathcal{F}, \mathcal{G}) = \sup \left\{ \frac{|P(A \cap B) - P(A)P(B)|}{P(A)^r P(B)^s} : A \in \mathcal{F}, B \in \mathcal{G} \text{ non-trivial} \right\}
\]

with the range of parameters \( 0 \leq r \leq 1, 0 \leq s \leq 1, r + s \leq 1 \). Clearly, \( \alpha_{r,s} \) is a number between 0 and 1. It is obvious that \( \alpha_{r,s} = 0 \) if and only if the \( \sigma \)-fields \( \mathcal{F}, \mathcal{G} \) are independent. Therefore one could use each of the coefficients \( \alpha_{r,s} \) as a measure of departure from independence.

Fortunately, among the infinite number of coefficients of dependence thus introduced, there are just four really distinct, namely \( \alpha_{0,0}, \alpha_{0,1}, \alpha_{1,0}, \) and \( \alpha_{1/2,1/2} \). By this we mean that the convergence to zero of \( \alpha_{r,s} \) (when the \( \sigma \)-fields \( \mathcal{F}, \mathcal{G} \) vary) is equivalent to the
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Convergence to 0 of one of the above four coefficients. And since \( \alpha_{0,1} \) and \( \alpha_{1,0} \) are mirror images of each other, we are actually left with three coefficients only.

The formal statement of this equivalence takes the form of the following inequalities.

**Proposition 6.1.1** If \( r + s < 1 \), then \( \alpha_{r,s} \leq (\alpha_{0,0})^{1-r-s} \).

If \( r + s = 1 \) and \( 0 < r \leq \frac{1}{2} \leq s < 1 \), then \( \alpha_{r,s} \leq (\alpha_{1/2,1/2})^{2r} \).

**Proof.** The first inequality follows from the fact that

\[
\frac{|P(A \cap B) - P(A)P(B)|}{P(A)^r P(B)^s} = |P(A \cap B) - P(A)P(B)|^{1-r-s}|P(B|A) - P(B)|^r |P(A|B) - P(A)|^s \\
\leq |P(A \cap B) - P(A)P(B)|^{1-r-s}.
\]

The second one is a consequence of

\[
\frac{|P(A \cap B) - P(A)P(B)|}{P(A)^r P(B)^s} = \left( \frac{|P(A \cap B) - P(A)P(B)|}{P(A)^{1/2} P(B)^{1/2}} \right)^{2r} |P(A|B) - P(A)|^{s-r} \leq (\alpha_{1/2,1/2})^{2r}.
\]

\[ \square \]

Coefficients \( \alpha_{0,0} \) and \( \alpha_{0,1}, \alpha_{1,0} \) are the basis for the definition of classes of stationary sequences called in the limit theorems literature strong-mixing and uniform strong mixing (called also \( \psi \)-mixing); \( \alpha_{1/2,1/2} \) is equivalent to the maximal correlation coefficient (6.3), which is the basis of the so called \( \rho \)-mixing condition. Monograph [39] gives recent exposition and relevant references; see also [42, pp. 380–385].

There is also a whole continuous spectrum of non-equivalent coefficients \( \alpha_{r,s} \) when \( r + s > 1 \). As those coefficients may attain value \( \infty \), they are less frequently used; one notable exception is \( \alpha_{1,1} \), which is the basis of the so called \( \psi \)-mixing condition and occurs occasionally in the assumptions of some limit theorems. Condition equivalent to \( \alpha_{1,1} < \infty \) and conditions related to \( \alpha_{r,s} \) with \( r + s > 1 \) are also employed in large deviation theorems, see [34, condition (U) and Chapter 5].

The following bounds\(^1\) for the covariances between random variables in \( L_p(F) \) and in \( L_q(F) \) will be used later on.

**Proposition 6.1.2** If \( X \) is \( F \)-measurable with \( p \)-th moment finite (\( 1 \leq p \leq \infty \)) and \( Y \) is \( G \)-measurable with \( q \)-th moment finite (\( 1 \leq q \leq \infty \)) and \( 1/p + 1/q \leq 1 \), then

\[
|EXY - EXEY| \leq 4(\alpha_{0,0})^{1-1/p-1/q}(\alpha_{1,0})^{1/p}(\alpha_{0,1})^{1/q}||X||_p||Y||_q \tag{6.1}
\]

where \( ||X||_p = (EX|X|^p)^{1/p} \) if \( p < \infty \) and \( ||X||_\infty = \text{ess sup}|X| \).

\(^1\)Similar results are also known for \( \alpha_{0,0} \) and \( \alpha_{1/2,1/2} \). The latter is more difficult and is due to R. Bradley, see [13, Theorem 2.2] and the references therein.