Some Remarks on Bezout's Theorem and Complexity Theory

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We begin by establishing the smoothness and irreducibility of certain algebraic varieties. Whereas these facts must be standard to algebraic geometers, they do not seem readily available.

For $d$ and $n$ positive integers, let $\mathcal{F}_{d,n}$ and $\mathcal{H}_{d,n}$ denote the spaces of polynomial mappings and homogeneous polynomial mappings $f: \mathbb{C}^n \to \mathbb{C}$ of degree less than or equal to $d$ in the case of $F_{d,n}$ and equal to $d$ in the case of $H_{d,n}$. For a multi-index $D = (d_1, \ldots, d_k)$, let $\mathcal{F}_{D,n}$ and $\mathcal{H}_{D,n}$ be the products

$$\prod_{i=1}^k \mathcal{F}_{d_i,n} \quad \text{and} \quad \prod_{i=1}^k \mathcal{H}_{d_i,n}.$$ 

So an element $F \in \mathcal{F}_{D,n}$ or $H_{D,n}$ is a polynomial mapping $F: \mathbb{C}^n \to \mathbb{C}^k$. The evaluation map $ev: \mathcal{F}_{D,n} \times \mathbb{C}^n \to \mathbb{C}^k$ and $ev: H_{D,n} \times \mathbb{C}^n \to \mathbb{C}^k$ is just the map $(F, x) \to F(x)$. For fixed $F$, $F^{-1}(0) \subset \mathbb{C}^n$ is the algebraic set determined by the simultaneous vanishing of the $f_{d_i,n}$.

**Lemma 1.** Any $y \in \mathbb{C}^k$ is a regular value for

$$ev: H_{D,n} \times (\mathbb{C}^n - \{0\}) \to \mathbb{C}^k$$

and

$$ev: F_{D,n} \times \mathbb{C}^n \to \mathbb{C}^k.$$

**Proof.** $Dev(F, x)(h, v) = h(x) + DF_x(v)$. The values of $h(x)$ alone are sufficient to make $D_{(F, x)}ev$ surjective.

Thus, by the implicit function theorem the union of the algebraic sets determined by the $F$'s is smooth in the product. For $F \in \mathcal{F}_{D,n}$, let $Z_F = \{x | F(x) = 0\}$, $Z_{F,0,n} = Z_F = \{(F, x) | F(x) = 0\}$. Similarly for $F \in H_{D,n}$, let $Z_F = \{x \in \mathbb{C}^n - \{0\} | F(x) = 0\}$, $Z_{F,0,n} = Z_F = \{(F, x) \in H_{D,n} \times (\mathbb{C}^n - \{0\}) | F(x) = 0\}$, and $Z_{F,0,n} = Z_F = \{(F, x) \in H_{D,n} | F \neq 0\}$; $Z_F$ and $Z_{F,0,n}$ are $ev^{-1}(0)$, so we have:

**Proposition 1.** (a) $Z_F$ is a connected smooth variety in $\mathcal{F}_{D,n} \times \mathbb{C}^n$ of codimension $k$;

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(b) $Z_{\chi}$ and $Z_{\chi'}$ are connected smooth varieties in $\mathcal{H}_{D,n} \times (\mathbb{C}^n - \{0\})$ of codimension $k$.

**Proof.** (b) It remains to prove the connectedness. The group of linear isomorphisms acts transitively on $\mathbb{C}^n - \{0\}$, and on $\mathcal{H}_{D,n} \times (\mathbb{C}^n - \{0\})$ by $(F, x) \rightarrow (F \circ L^{-1}, Lx)$, this action preserves $Z_{\chi}$ and $Z_{\chi'}$. Thus, the maps $Z_{\chi} \rightarrow \mathbb{C}^n - \{0\}$, $Z_{\chi'} \rightarrow \mathbb{C}^n - \{0\}$ are surjective locally trivial fibrations with connected base and connected fiber so they are connected, as follow: Given $(f_1, x_1)$ and $(f_2, x_2)$ in $Z_{\chi'}$, choose a path $x_t$ from $x_1$ to $x_2$, for $1 \leq t \leq 2$. Now lift $x_t$ to $(\hat{f}_t, x_t)$ such that $\hat{f}_1 = f_1$; the endpoint of this path is in the fiber over $x_2$. This is a complex linear space minus 0 and, hence, is connected, so we continue the path in the fiber to $(f_2, x_2)$. The same argument holds for $Z_{\chi}$; for $Z_{\chi'}$, simply replace the linear group by the affine group.

We use $P(V)$ to denote the projective space of the vector space $V$, i.e., $V - 0$ mod the action of the nonzero scalars $\mathbb{C}^* = \mathbb{C} - 0$ and $PC(n - 1)$ for the projective space of $\mathbb{C}^n$.

$\mathbb{C}^* \times \mathbb{C}^*$ acts freely on $(\mathcal{H}_{D,n} - \{0\}) \times (\mathbb{C}^n - \{0\})$ by coordinatewise multiplication. Let $N$ denote the dimension of $\mathcal{H}_{D,n}$:

$$N = \sum_{i=1}^k \left( n + d_i - 1 \right).$$

The $\mathbb{C}^* \times \mathbb{C}^*$ action leaves $Z_{\chi'} \subset \mathcal{H}_{D,n} \times \mathbb{C}^n - \{0\}$ invariant. As the action is transversal to $S^{2N-1} \times S^{2n-1}$, $Z_{\chi'} \cap S^{2N-1} \times S^{2n-1}$ is a smooth manifold, and, therefore, the quotient of $Z_{\chi'}$ by the $\mathbb{C}^* \times \mathbb{C}^*$ action is the same as $Z_{\chi'} \cap S^{2N-1} \times S^{2n-1}$ by the unit complexes $S^1 \times S^1$. This later group is compact. So the quotient by the free action is a smooth subvariety $\mathcal{Z}_{D,n} = \mathcal{Z}$ of $P(\mathcal{H}_{D,n}) \times PC(n - 1)$. As $Z_{\chi'}$ is connected, so is the quotient manifold $\mathcal{Z}$. A connected, smooth projective variety is irreducible.

**Theorem 1.** $\mathcal{Z}_{D,n}$ is a connected, smooth irreducible projective subvariety of $P(\mathcal{H}_{D,n}) \times PC(n - 1)$ of codimension $k$.

Let $C_{D,n} = \{ F \in \mathcal{H}_{D,n} - \{0\} | \exists x \in \mathbb{C}^n - \{0\} \text{ with } F(x) = 0 \}$, i.e., $C_{D,n}$ is the set of those systems with a common root. Let $\mathcal{C}_{D,n}$ be the image of $C_{D,n}$ in $P(\mathcal{H}_{D,n})$.

**Corollary 1.** $\mathcal{C}_{D,n}$ is an irreducible subvariety of $P(\mathcal{H}_{D,n})$.

**Proof.** It is the projection of $\mathcal{Z}_{D,n}$ on $P(\mathcal{H}_{D,n})$; as $\mathcal{Z}_{D,n}$ is irreducible, its image must be.

The case of $(n - 1)$ homogeneous polynomials in $n$ variables is the case of Bezout's theorem; there are generically...