Let $p^D_t(x, y) = p_t(x, y)$ be the Dirichlet heat kernel for $\frac{1}{2} \Delta$ in a domain $D \subset \mathbb{R}^n$, $n \geq 2$. In [4] E.B. Davies and B. Simon define the semigroup connected with the Dirichlet Laplacian to be intrinsically ultracontractive if there is a positive (in $D$) eigenfunction $\phi_0$ for $\frac{1}{2} \Delta$ in $D$ and if for each $t > 0$ there are positive constants $c_t$, $C_t$ depending only on $D$ and $t$ such that

$$c_t \phi_0(x) \phi_0(y) < p_t(x, y) < C_t \phi_0(x) \phi_0(y), \quad x, y \in D.$$  

(To be precise, they show (1) is equivalent to their definition.) Here we will say $D$ is IU when (1) holds. Among many results in their very interesting paper Davies and Simon prove that bounded Lipschitz domains are IU. Recently the investigation of IU has been taken up by a number of probabilists who approached the area via their study of distributions of the lifetimes of Doob's conditioned h-processes, a study which in its modern version started with Cranston and McConnell's solution ([3]) of a conjecture of Chung. Here we are going to give an intuitive sketch of the connection between lifetimes and IU, and describe the results in our paper [6].

The reader will note that Rodrigo Banuelos's paper in this volume deals with the same topic as this paper. Bañuelos was more diligent than we were and wrote his paper first. Sometimes sloth is rewarded (besides being its own reward), and this is one of those times, for, since Bañuelos has surveyed the area, we do not have to. Accordingly only those papers with the most immediate connection to [6] will be mentioned.

Let $X = \{X_t, t \geq 0\}$, be standard $n$ dimensional Brownian motion and let $p^x$ stand for probability associated with this motion given $X_0 = x$. The kernel $p_t(x, \cdot)$ has an immediate probabilistic interpretation as the density at time $t$ of $X$ killed when it leaves $D$, that is, if $A$ is a Borel subset of $D$ and $\tau_D = \inf\{t > 0 : X_t \notin D\}$,

$$P_x(X_t \in A, t < \tau_D) = \int_A p_t(x, y) dy.$$ 

Thus $p_t(x, y)/p_s(x, y)$ gives the ratio of the probabilities of killed Brownian motion being infinitessimally close to $y$ (we will just say "hitting $y" ") at times $t$ and $s$, respectively.

Now it can be shown that for $\delta > 0$ fixed,

$$\lim_{\epsilon \to 0} \frac{P_x(\exists s, t < \tau_D : |s - t| > \delta \quad \text{and} \quad |B_s - y| < \epsilon, |B_t - y| < \epsilon)}{P_x(\exists t < \tau_D : |B_t - y| < \epsilon)} = 0,$$

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and roughly what this implies is that given \( y \) is hit by \( X \) before \( \tau_D \), it is hit at only one time, although some work must be done to make this rigorous, since the probability \( X \) ever hits \( y \) is zero. Thus intuitively, \( p_t(x,y)/p_s(x,y) \) gives the ratio of the probabilities of \( X \) first hitting \( y \) before \( \tau_D \) at times \( t \) and \( s \), respectively, if \( X_0 = x \), given \( X \) does hit \( y \) before time \( \tau_D \). Equivalently,

\[
L_{x,y}(t) = \frac{p_t(x,y)}{\int_0^\infty p_t(x,y)dt}
\]

is the density of the time \( X \) (started at \( x \)) first hits \( y \), given that \( y \) is hit before \( \tau_D \). The integral in the denominator is finite whenever \( x \neq y \) and \( D \) has a Green function, assumptions always in force here. We use \( P^n_x \) and \( E^n_x \) to denote probability and expectation associated with \( X \), started at \( x \), conditioned to hit \( y \) before \( \tau_D \), and use \( T \) to denote the time of hitting \( y \), so that the density of \( T \) under \( P^n_x \) is \( L_{x,y}(\cdot) \). Note \( E^n_x T = \int_0^\infty tL_{x,y}(t)dt \).

Cranston and McConnel's theorem is equivalent to the following.

**Theorem.** Let \( n = 2 \). There is an absolute constant \( C \) such that

\[
\int_0^\infty tL_{x,y}(t)dt < C \text{ area } D, \ x \neq y, \ x, y \in D.
\]

Thus the Cranston–McConnell theorem can be interpreted as as theorem about the shape of the normalized (in \( t \)) heat kernel which holds uniformly over all points \( x, y \). We will call domains satisfying \( \sup_{x,y\in D} E^n_x T < \infty \) Cranston–McConnell domains. Let

\[
\hat{p}_t(x, \cdot) = p_t(x, \cdot)/\int_{\{y \neq x, y \in D\}} pt(x,y)dy.
\]

It is shown in [6] that \( D \) is IU if and only if there is for each \( t > 0 \) a positive constant \( \alpha_t \) such that \( \hat{p}_t(x,y)/\hat{p}_t(x,y) < \alpha_t \), \( x, y, z \in D \). The only if part is immediate, and the if part is easy. Thus IU is equivalent to a different kind of uniformity of the heat kernel, uniformity in the first variable under normalization in the second variable. Since \( \hat{p}_t(x, y) \) is the density of killed Brownian motion, started at \( x \), conditioned to be alive at time \( t \), IU says all such densities are comparable, so it is in some sense a mixing condition for this motion.

Now it is immediate that if \( \lambda \) is the eigenvalue associated with the eigenfunction \( \phi_0 \), both \( c_1 e^{-\lambda t} \) and \( C_1 e^{-\lambda t} \) may be chosen to be respectively nondecreasing and nonincreasing in \( t \), since

\[
\int_D \phi_0(y)p_s(y,z)dy = e^{-\lambda s}\phi_0(z),
\]

and

\[
\int_D p_t(x,y)p_s(y,z)dy = p_{s+t}(x,z).
\]