CHAPTER 8

Pipes and Horns

The wave propagation phenomena in fluids that we have examined in previous chapters have referred to waves in infinite or semi-infinite spaces generated by the vibrational motion of some small object or surface in that space. We now turn to the very different problem of studying the sound field inside the tube of a wind instrument. Ultimately, we shall join together the two discussions by considering the sound radiated from the open end or finger holes of the instrument, but for the moment our concern is with the internal field. We begin with the very simplest cases and then add complications until we have a reasonably complete representation of an actual instrument. At this stage, we will find it necessary to make a digression, for a wind instrument is not excited by a simple source, such as a loudspeaker, but is coupled to a complex pressure-controlled or velocity-controlled generator—the reed or air jet—and we must understand the functioning of this before we can proceed. Finally, we go on to treat the strongly coupled pipe and generator system that makes up the instrument as played.

8.1. Infinite Cylindrical Pipes

The simplest possible system of enclosure is an infinite cylindrical pipe or tube with its axis parallel to the direction of propagation of a plane wave in the medium (Morse and Ingard, 1968). If the walls of the pipe are rigid, perfectly smooth, and thermally insulating, then the presence of the tube wall has no effect on wave propagation. A pressure wave propagating in the x direction has the form

\[ p(x,t) = p \exp[j(-kx + \omega t)], \] (8.1)

and the resultant acoustic volume flow is, as we saw in Chapter 6,

\[ U(x,t) = \left( \frac{Sp}{\rho c} \right) \exp[j(-kx + \omega t)], \] (8.2)

where \( \omega \) is the angular frequency, \( k \) is the angular wave number \( k = 2\pi/\lambda = \omega/c \), and \( S \) is the cross-sectional area of the pipe. As usual, \( \rho \) is the density of and
8.1. Infinite Cylindrical Pipes

$c$ the velocity of sound in air. The acoustic impedance of the pipe at any point $x$ is

$$Z_0(x) = \frac{p(x, t)}{U(x, t)} = \frac{\rho c}{S}.$$  \hspace{1cm} (8.3)

To treat this problem in more detail, we must solve the wave equation directly in cylindrical polar coordinates $(r, \phi, x)$. If $a$ is the radius of the pipe and its surface is again taken to be perfectly rigid, then the boundary condition is

$$\frac{\partial p}{\partial r} = 0 \quad \text{at} \quad r = a,$$  \hspace{1cm} (8.4)

which implies that there is no net force and therefore no flow normal to the wall. The wave equation in cylindrical coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2},$$  \hspace{1cm} (8.5)

and this has solutions of the form

$$p_{mn}(r, \phi, x) = p_{\text{sin}}^\text{cos}(m\phi) J_m \left( \frac{\pi q_{mn}}{a} \right) \exp[j(-k_{mn}x + \omega t)],$$  \hspace{1cm} (8.6)

where $J_m$ is a Bessel function and $q_{mn}$ is defined by the boundary condition [Eq. (8.4)], so that the derivative $J'_m(\pi q_{mn})$ is zero. The $(m, n)$ mode thus has an $(r, \phi)$ pattern for the acoustic pressure $p$ with $n$ nodal circles and $m$ nodal diameters, both $m$ and $n$ running through the integers from zero. In the full three-dimensional picture, these become nodal cylinders parallel to the axis and nodal planes through the axis, respectively.

In Fig. 8.1, the pressure and flow velocity patterns for the lowest three modes of the pipe, omitting the simple plane-wave mode, are shown. The

![Fig. 8.1. Pressure and transverse flow patterns for the lowest three transverse modes of a cylindrical pipe. The plane-wave mode is not shown.](image-url)