CHAPTER 4

ELEMENTARY C*-ALGEBRA THEORY

4.6. Exercises

4.6.1. Suppose that \(S_1, S_2, T_1, T_2,\) and \(A\) are elements of a C*-algebra \(\mathfrak{A},\) and

\[
0 \leq S_1 \leq T_1, \quad 0 \leq S_2 \leq T_2.
\]

Prove that

\[
\|S_1^{1/2}A\| \leq \|T_1^{1/2}A\|,
\]

and deduce that

\[
\|S_1^{1/2}AS_2^{1/2}\| \leq \|T_1^{1/2}AT_2^{1/2}\|.
\]

Solution. Since

\[
0 \leq (S_1^{1/2}A)^*(S_1^{1/2}A) = A*S_1A \leq A*T_1A = (T_1^{1/2}A)^*(T_1^{1/2}A),
\]

we have

\[
\|S_1^{1/2}A\| = \|A*S_1A\|^{1/2} \leq \|A*T_1A\|^{1/2} = \|T_1^{1/2}A\|.
\]

A similar argument shows that

\[
\|S_2^{1/2}B^*\| \leq \|T_2^{1/2}B^*\|,
\]
equivalently

\[
\|BS_2^{1/2}\| \leq \|BT_2^{1/2}\|,
\]

for \(B\) in \(\mathfrak{A}.\) Upon replacing \(A\) by \(AS_2^{1/2}\) and \(B\) by \(T_1^{1/2}A,\) we obtain

\[
\|S_1^{1/2}AS_2^{1/2}\| \leq \|T_1^{1/2}AS_2^{1/2}\| \leq \|T_1^{1/2}AT_2^{1/2}\|.
\]
4.6.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras and $\varphi$ is a * homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Suppose that $B, K \in \mathcal{B}$, with $B$ self-adjoint and $K$ positive, and let $V$ be the exponential unitary $\exp iB$. Show that there exist $A, H, U \in \mathcal{A}$, with $A$ self-adjoint and $H$ positive, such that

$$
\varphi(A) = B, \quad \varphi(H) = K, \quad \varphi(U) = V.
$$

[See also Exercises 4.6.3 and 4.6.59.]

**Solution.** Since $\varphi(\mathcal{A}) = B$, we have $B = \varphi(A_0)$ for some $A_0$ in $\mathcal{A}$. Since

$$
\varphi(A_0^*) = \varphi(A_0)^* = B^* = B,
$$

we can conclude that $\varphi(A) = B$, where $A$ is the self-adjoint element $\frac{1}{2}(A_0 + A_0^*)$ of $\mathcal{A}$. By Theorem 4.1.8(ii)

$$
V = \exp(iB) = \exp i\varphi(A) = \varphi(\exp iA).
$$

With $K$ in place of $B$, the above argument shows that $K = \varphi(S)$ for some self-adjoint $S$ in $\mathcal{A}$. With $f : \mathbb{R} \to \mathbb{R}$ the continuous function defined by $f(t) = \max\{t, 0\}$, we have $f(K) = K$, while $f(S)$ is a positive element $H (= S^+)$ of $\mathcal{A}$. Again by Theorem 4.1.8(ii),

$$
K = f(K) = f(\varphi(S)) = \varphi(f(S)) = \varphi(H). \quad \blacksquare
$$

4.6.3. Suppose that $\mathbb{D}$ is the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$, $\mathbb{T}$ is its boundary $\{z \in \mathbb{C} : |z| = 1\}$, and $\mathbb{S}$ is the union $\{\exp i\theta : \theta \in \mathbb{R}, \pi/4 \leq |\theta| \leq 3\pi/4\}$ of two closed arcs in $\mathbb{T}$. Consider the $C^*$-algebras $C(\mathbb{D}), C(\mathbb{T}), C(\mathbb{S})$ and the * homomorphisms $\varphi$ (from $C(\mathbb{D})$ onto $C(\mathbb{T})$) and $\psi$ (from $C(\mathbb{T})$ onto $C(\mathbb{S})$) defined by restriction; that is,

$$
\varphi(f) = f|\mathbb{T}, \quad \psi(g) = g|\mathbb{S} \quad (f \in C(\mathbb{D}), \ g \in C(\mathbb{T})).
$$

Find

(i) a unitary element $u$ of $C(\mathbb{T})$ that is not of the form $\varphi(f)$ for any invertible element $f$ of $C(\mathbb{D})$; [Hint. Use the fact (from elementary algebraic topology) that there is no continuous mapping of $\mathbb{D}$ onto $\mathbb{T}$ that leaves each point of $\mathbb{T}$ fixed—that is, $\mathbb{T}$ is not a retract of $\mathbb{D}$.]

(ii) a projection $q$ in $C(\mathbb{S})$ that is not of the form $\psi(p)$ for any projection $p$ in $C(\mathbb{T})$.

[See also Exercise 4.6.59.]