Group Symmetries of Antenna Arrays

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9.1. Introduction

The analysis of the radiation patterns of three-dimensional arrays has, up to the present time, been limited to a few well-known structures which were introduced as natural extensions of simple geometric shapes, such as spheres or cylinders, etc. At the present time no systematic procedure exists for choosing the proper three-dimensional arrangement of radiators in space. New geometrical concepts are needed to enrich the presently available techniques of analysis of antenna theory. An area where similar geometrical concepts have been developed with great success is crystallography. In this area, using group theory as the underlined mathematical structure, a large class of crystal lattices in space were studied and a very elegant and systematic formalism was developed for the classification of crystal structures.

In this chapter a new class of three-dimensional lattices will be studied and a number of their radiation properties will be examined. The basic formalism of treating these problems has been transplanted from crystallography and solid state theory. In examining the properties of the various crystal lattices, group theory is used as a powerful tool for the systematic analysis of the various three-dimensional structures and their associated wave functions. In an analogous manner, group theory will be used for the systematic classification of array lattices and their associated radiation patterns.

The groups which are of interest in this work are those whose elements correspond to certain operations in space. These operations are translation, reflection, rotation, and inversion. An example of these groups is the full rotation group $O_3$, which includes infinitesimal rotations plus coordinate inversions. A number of investigators have used the properties of the $O_3$ group to examine the symmetries of electromagnetic fields. Liubarskii [1], Tinkham [2], Gelfand [3], and Naimark [4] have studied the rotational properties of

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The groups which are pertinent to the development of the present work are the finite rotational groups which are known as the crystallographic point groups. The point groups are a subgroup of the rotation group $O_3$ and were successfully employed in the classification of crystal lattices. This formalism can be used very effectively for the systematic classification of array lattices. Alternative geometries can be examined and the global properties of the radiation patterns can be analyzed without having to carry out extensive integrations of the radiation integrals. It is also a powerful tool that can be used to design new types of antennas whose radiation fields are orthogonal, thus introducing a new synthesis procedure.

9.2. Group Theory Fundamentals

For the sake of completeness a number of basic properties of groups will be introduced here to lay the groundwork for the following discussion.

**Group.** A group is a set of elements $g_i \in G$ for which an operation is defined with the properties:

(a) $g_i g_j = g_k$, $g_k \in G$.
(b) Associativity, $g_i (g_j g_k) = (g_i g_j) g_k$.
(c) There exists an identity element $E$; $E g_j = g_j = g_j E$.
(d) For every element $g_i$ there exists an inverse element $g_i^{-1}$ such that $g_i^{-1} g_i = E$.

**Representation of a Group.** For each element $g_i$ we associate a matrix $\Gamma_{\mu \nu}^{(i)}(g_i)$ such that it has the same operational properties under matrix multiplication as the elements of the group

$$g_i g_j = g_k \quad \Rightarrow \quad \Gamma_{\mu \nu}^{(i)}(g_i) \Gamma_{\kappa \lambda}^{(j)}(g_j) = \Gamma_{\mu \nu}^{(k)}(g_k). \quad (9.1)$$

**The Irreducible Representation.** For a given element of a group there exists a multiplicity of matrix representations obtained by similarity transformations. The representation of the lowest dimensionality in a block form is called irreducible.

**Character of a Representation.** It is defined as $\chi(g_i) = \text{Tr} \ \Gamma_{\mu \nu}^{(i)}(g_i)$. The character is a constant and is the same for all representations within the same class.

**The basis Functions.** A set of functions $\varphi_{\nu}$ exists such that for every operation they transform as

$$\varphi_{\mu}^{(i)} = \Gamma_{\mu \nu}^{(i)}(g_i) \varphi_{\nu}^{(i)}. \quad (9.2)$$