On Stabilization and the Existence of Coprime Factorizations
Malcolm C. Smith

Abstract

We show that any transfer function matrix whose elements belong to the quotient field of $H_{\infty}$, and which is stabilizable, has a matrix fraction representation over $H_{\infty}$ which is coprime in the sense that a matrix Bezout identity can be satisfied.

1. Introduction

The following question was posed by Vidyasagar, Schneider and Francis [9]. Let $\mathcal{R}$ be an integral domain and $\mathcal{F}$ its quotient field. Let $P \in \mathcal{F}^{n \times m}$ and suppose that $C \in \mathcal{F}^{m \times n}$

\[
\begin{bmatrix}
(I + CP)^{-1} & -C(I + PC)^{-1} \\
P(I + CP)^{-1} & (I + PC)^{-1}
\end{bmatrix}
\]

has all its elements in $\mathcal{R}$. Is it true that $P$ has \textit{strongly coprime} right and left factorizations over $\mathcal{R}$, i.e. do there exist matrices $N, D, X, Y, \hat{N}, \hat{D}, \hat{X}, \hat{Y}$ with elements in $\mathcal{R}$ and of appropriate dimension such that $P = ND^{-1} = \hat{D}^{-1}\hat{N}$ and

\[YD + XN = I, \quad \hat{D}\hat{Y} + \hat{N}\hat{X} = I?\]

In the case where $\mathcal{R}$ is a Bezout domain the answer is yes (see [9]) and this covers many
cases of interest in system theory. On the other hand it has been shown by Anantharam [1] that the conclusion fails for general integral domains \( \mathcal{R} \). A counterexample was presented for the case of \( \mathcal{R} = \mathbb{Z}[\sqrt{-5}] \). For other choices of \( \mathcal{R} \) the question has remained open. The main purpose of this paper is to show (Theorem 1) that the answer to the above question is yes for the important case of \( \mathcal{R} = H_\infty \). Let the quotient field of \( H_\infty \) be denoted by \( F_\infty \).

**Theorem 1.** Suppose \( P \in F_\infty^{n \times m} \) is stabilizable. Then \( P \) has strongly coprime right and left factorizations over \( H_\infty \).

This result means that the full power of the fractional representation theory can be applied to any stabilizable plant whose transfer function matrix has elements in the field \( F_\infty \). In particular, the set of all stabilizing compensators can be parametrized in the standard way.

Our proof of Theorem 1 involves showing that any pair of \( H_\infty \) functions possesses a greatest common divisor (Theorem 2). This fact appears only to be widely known for the special case of a pair of inner functions. In Section 2 we show that this result leads to an easy proof of Theorem 1 in the scalar case. The generalization to the matrix case (Section 3) involves showing that a square right (left) matrix factor can be extracted from a “tall” (“wide”) matrix over \( H_\infty \) so that the highest order minors have no non-trivial common divisor (Lemma 5). In Lemma 4 we establish a general fact about matrices over rings (with the property that greatest common divisors exist) which allows the algebraic nature of the scalar proof to be retained in the matrix case.

The results of this paper appeared independently in [5] and [7]. The approach of Inouye builds on previous work of the same author and exploits a result of Helson-Lowdenslager concerning matrix inner coprimeness. The approach given here is different from [5]. The contributions of this paper include (1) establishing the existence of greatest common divisors for \( H_\infty \) functions and (2) providing an approach (using Lemma 4) for answering the question of Vidyasagar, Schneider and Francis for other integral domains \( \mathcal{R} \).

2. The Scalar Case

We begin with some terminology concerning integral domains. A *greatest common divisor* (gcd) of a set of elements of \( \mathcal{R} \) is a common divisor which is a multiple of any other common divisor. A gcd is unique up to multiplication by an invertible element of \( \mathcal{R} \). We will use the notation \( a = \gcd(r_1, \ldots, r_n) \) to mean that \( a \in \mathcal{R} \) is a gcd of \( r_1, \ldots, r_n \in \mathcal{R} \). We define \( a, b \in \mathcal{R} \) to be *weakly coprime* if \( 1 = \gcd(a, b) \) and *strongly coprime* if there exist \( x, y \in \mathcal{R} \) such that \( ax + by = 1 \). We say \( \mathcal{R} \) is a *greatest common divisor domain* (GCDD) if any pair (and hence any finite set) of elements has a gcd. We say \( \mathcal{R} \) is a *Bezout domain* (BD) if every finitely generated ideal of \( \mathcal{R} \) is principal. A necessary and