Abstract

We present in this paper a fairly general mathematical analysis of the vortex method of approximation of the Euler equations for an incompressible fluid flow. We also discuss some recent methods of numerical approximation of viscous terms using the vortex method.

1. Introduction

The numerical simulation of incompressible fluid flows at high Reynolds numbers is a challenging problem in the sense that, in many flows of practical importance, the vorticity has a singular behaviour and is concentrated in thin vortex sheets which move with the fluid. Clearly, conventional methods such as finite-difference or finite-element methods or even spectral methods are not well suited for computing such flows. On the other hand, in vortex methods, the vorticity field is approximated by a linear combination of Dirac measures (the vortex particles) whose motions are Lagrangian, i.e., the Lagrangian
coordinates of the vortex particles are fixed. In fact, vortex methods appear to be well adapted to the numerical simulation of the singular flows mentioned above and have been widely used in a number of applications. However, the mathematical analysis of vortex methods is far more involved than that of conventional methods and yields a number of interesting and difficult open questions.

The purpose of this paper is to present together with the companion paper of J.T. Beale an overview of the available mathematical theory of the vortex method. More precisely, we shall introduce on the one hand a fairly general mathematical analysis of the vortex methods of approximation of the Euler equations. We shall review on the other hand some recent methods of numerical approximation of the viscous terms of the Navier-Stokes equations.

We shall not consider here the numerical treatment of boundary conditions by the vortex method but we want to underline that this is indeed a very important problem which needs further investigation.

2. Convergence of the vortex method

Consider the two-dimensional Euler equations for an incompressible fluid flow in the whole plane:

\[
\begin{align*}
\frac{\partial}{\partial t} u + (u \cdot \nabla) u + \nabla p &= f \\
\nabla \cdot u &= 0
\end{align*}
\quad \forall \xi \in \mathbb{R}^2 \, , \, t > 0
\]

with the condition at infinity

\[
u(x,t) \rightarrow \nu_\infty \quad \text{as} \quad |\xi| \rightarrow +\infty
\]

and the initial condition

\[
u(\xi,0) = \nu_0 \quad .
\]

In the above equations, \( \nu = (u_1, u_2) \) is the fluid velocity, \( p \) its pressure and \( f \) represents the body forces.

Assume that the forces \( f \) are potential and introduce the vorticity

\[
\omega = \text{curl} \, \nu = \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right). \quad \text{Then, the Euler equations can be equivalently written}
\]