3

Formal Language and Proof

3.1 Propaganda

We finally turn our attention to quantifiers, which we will discuss at rather a formal level. But the use of quantifiers and proofs involving them is part of the larger discussion about why mathematical proofs are hard to read or write. The reason is that mathematical proofs (at least those written in paragraph form) are written in a mixture of two languages. The last chapter was all about the first language, which is essentially bits of ordinary English borrowed to produce the paragraph; frequently this part of the language serves to indicate the structure of the proof. But the structure of the proof is dictated by the second language, which is pieces of formal logic borrowed and adapted for mathematics.\(^1\) A good deal of the difficulty with writing mathematics is that most people don’t know the formal language of mathematical logic, and so have to pick up the borrowed parts bit by bit. It’s a little as if you were trying to learn how to write in a mixture of French and Spanish, and, while you were fluent in French, all you knew about Spanish was the bits you had seen other people write in the mixture, but people expected you to write as if you were fluent in Spanish as well.

One solution would be simply to take a course in mathematical logic, but that’s a large investment in time, and most mathematicians don’t do it. What we will do here is to try to give you enough grounding in logic

\(^{1}\)If you think about it, the discussion of proof structures in Section 2.3 was really about the easy part of this formal language.
so that you can work with quantifiers in proofs. Along the way we will see 
some overlap of mathematical logic (the formal language) with the informal 
and be able to state precisely some things we rather talked around in the 
previous chapter. The goal is not to make you a mathematical logician; the 
goal is to make you comfortable enough with quantifiers.

Why does anyone bother with formal logical language anyway? It has numerous disadvantages: it is very formal, with tightly prescribed rules of 
grammar and manipulation. It is quite limited, with a rather small vocabulary. The things it can talk about form a very small part of the possible 
objects of human thought and so it is nowhere near as rich as a natural 
language such as English, Japanese, or Hausa (you can’t, for example, talk 
about poetry, love, or the weather). But the payoff for the restrictions and 
high degree of structure is great efficiency (a mathematical sentence may 
take pages of English to translate) and great precision (lack of ambiguity). 
That precision is really required to do mathematics.

With this going for it, why is formal mathematical language feared and 
loathed by so many students, particularly those who are having to grapple 
with writing proofs? The flip side of “efficient” is “intimidating, densely 
written, and hard to read”: the tightly prescribed “rules” are easy to violate, 
which makes it hard to write. The language of mathematics just doesn’t 
seem very user-friendly, and, since you get to see only its “bits and pieces,” 
nobody gives you a fair chance to learn it in the first place.

The above negatives are partly true, but the requirement for precision 
overrides them. The goal of this chapter therefore is, first, to give you a fair 
chance to learn formal mathematical language and, second, to convince you 
that a sensitivity to its use can make your life as a prover of mathematics 
much easier. In particular, the way it is written can help you discover 
proofs, if only you learn the clues hidden in the writing.

Let’s give an example of how sensitivity to formal structure is useful. 
Consider the following:

\[ 5 = 4 \iff 5 + \ldots \]

You see instantly that this mish-mash makes no sense for “grammatical” or “structural” reasons. The problem isn’t the meaning of the symbols 
(“5” still means what it always does), it is that they aren’t put together 
correctly. For a more positive example, consider the following incomplete 
expression:

\[ f(x) = a_{17}x^{17} + a_{16}x^{16} + \ldots \]

which you are indeed likely to think incomplete. What might come after 
the \ldots? Lots of things might, but you unconsciously rule out many others. 
Your familiarity with the language of mathematics gives you an expectation

\[ \text{\url{2}} \text{There’s always a professor waiting to pounce on the slightest error as if it had earthshaking consequences.} \]