Infinite and Infinitesimal Numbers

Sierpiński’s Luggage

Wacław Sierpiński, the great Polish mathematician, was very interested in infinite numbers. The story, presumably apocryphal, is that once when he was travelling, he was worried that he’d lost one piece of his luggage. “No, dear!” said his wife, “All six pieces are here.” “That can’t be true,” said Sierpiński, “I’ve counted them several times: zero, one, two, three, four, five.”

Counting from Zero

Many mathematicians prefer to count objects starting at “zero” rather than “one.” Although this may be unfamiliar, it is really a much simpler method. In fact, unless we teach them otherwise, machines tend to do it without thinking: One thousand cloakroom tickets will probably be numbered 000 to 999. If you count this way, then the number of objects you’ve counted is the earliest number that you didn’t use, rather than the latest one you did (see Figure 10.1).
"zero, one, two, three, four, five, ... so there are SIX bags here."

**FIGURE 10.1** What Sierpiński should have said.

### The Empty Set

One of the advantages of the new system is that it works even when you are counting no objects at all. If Sierpiński's luggage all gets lost en route, then, at the other end of his journey he should say:

"...", so there are ZERO bags here!

The usual system of counting doesn't work for counting zero objects, since there isn't a last number that you used.

### Cantor's Ordinal Numbers

The great German mathematician Georg Cantor was the earliest person to construct a coherent theory of counting collections that may be infinite. For this he extended the ordinary series of numbers used for counting, as follows:

\[
0, 1, 2, \ldots \text{ as usual,} \\
\text{then } \omega, \, \omega+1, \, \omega+2, \ldots \text{ then } \omega+\omega, \, \omega+\omega+1, \ldots
\]

and so on.

The important point about these numbers (and, in essence, their definition) is that, no matter how many of them you've used, there's always a (uniquely determined) earliest one that you haven't. Cantor's opening infinite number,

\[
\omega = \{0, 1, 2, \ldots \} 
\]

is defined to be the earliest number greater than all the finite counting numbers. We'll use

\[
\{a, b, c, \ldots \}
\]