CHAPTER 12

The Word Problem

Introduction

Novikov, Boone, and Britton proved, independently, that there is a finitely presentable group $G$ for which no computer can ever exist that can decide whether an arbitrary word on the generators of $G$ is 1. We shall prove this remarkable result in this chapter.

Informally, if $L$ is a list of questions, then a decision process (or algorithm) for $L$ is a uniform set of directions which, when applied to any of the questions in $L$, gives the correct answer "yes" or "no" after a finite number of steps, never at any stage of the process leaving the user in doubt as to what to do next.

Suppose now that $G$ is a finitely generated group with the presentation

$$G = \langle x_1, \ldots, x_n | r_j = 1, j \geq 1 \rangle;$$

every (not necessarily reduced) word $\omega$ on $X = \{x_1, \ldots, x_n\}$ determines an element of $G$ (namely, $\omega R$, where $F$ is the free group with basis $X$ and $R$ is the normal subgroup of $F$ generated by $\{r_j, j \geq 1\}$). We say that $G$ has a solvable word problem if there exists a decision process for the set $L$ of all questions of the form: If $\omega$ is a word on $X$, is $\omega = 1$ in $G$? (It appears that solvability of the word problem depends on the presentation. However, it can be shown that if $G$ is finitely generated and if its word problem is solvable for one presentation, then it is solvable for every presentation with a finite number of generators.)

Arrange all the words on $\{x_1, \ldots, x_n\}$ in a list as follows: Recall that the length of a (not necessarily reduced) word $\omega = x_1^{e_1} \cdots x_n^{e_m}$, where $e_i = \pm 1$, is $m$. For example, the empty word $1$ has length 0, but the word $xx^{-1}$ has length 2. Now list all the words on $X$ as follows: first the empty word, then the
words of length 1 in the order $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$, then the words of length 2 in “lexicographic” order (as in a dictionary): $x_1 x_1 < x_1 x_1^{-1} < x_1 x_2 < \cdots < x_1^{-1} x_1 < x_1^{-1} x_1^{-1} < \cdots < x_n^{-1} x_n$, then the words of length 3 in lexicographic order, and so forth. Use this ordering of words: $\omega_0, \omega_1, \omega_2, \ldots$ to define the list $\mathcal{L}$ whose $k$th question asks whether $\omega_k = 1$ in $G$.

We illustrate by sketching a proof that a free group $G = (x_1, \ldots, x_n | \emptyset)$ has a solvable word problem. Here is a decision process.

1. If $\text{length}(\omega_k) = 0$ or 1, proceed to Step 3. If $\text{length}(\omega_k) \geq 2$, underline the first adjacent pair of letters, if any, of the form $x_i x_i^{-1}$ or $x_i^{-1} x_i$; if there is no such pair, underline the final two letters; proceed to Step 2.
2. If the underlined pair of letters has the form $x_i x_i^{-1}$ or $x_i^{-1} x_i$, erase it and proceed to Step 1; otherwise, proceed to Step 3.
3. If the word is empty, write $\omega_k = 1$ and stop; if the word is not empty, write $\omega_k \neq 1$ and stop.

The reader should agree, even without a formal definition, that the set of directions above is a decision process showing that the free group $G$ has a solvable word problem.

The proof of the Novikov–Boone–Britton theorem can be split in half. The initial portion is really Mathematical Logic, and it is a theorem, proved independently by Markov and Post, that there exists a finitely presented semigroup $S$ having an unsolvable word problem. The more difficult portion of the proof consists of constructing a finitely presented group $\mathcal{B}$ and showing that if $\mathcal{B}$ had a solvable word problem, then $S$ would have a solvable word problem. Nowhere in the reduction of the group problem to the semigroup problem is a technical definition of a solvable word problem used, so that the reader knowing only our informal discussion above can follow this part of the proof. Nevertheless, we do include a precise definition below. There are several good reasons for doing so: the word problem can be properly stated; a proof of the Markov–Post theorem can be given (and so the generators and relations of the Markov–Post semigroup can be understood); a beautiful theorem of G. Higman (characterizing the finitely generated subgroups of finitely presented groups) can be given. Here are two interesting consequences: Theorem 12.30 (Boone–Higman): there is a purely algebraic characterization of groups having a solvable word problem; Theorem 12.32 (Adian–Rabin): given almost any interesting property $P$, there is no decision process which can decide, given an arbitrary finite presentation, whether or not the presented group enjoys $P$.

**Exercises**

12.1. Sketch a proof that every finite group has a solvable word problem.

12.2. Sketch a proof that every finitely generated abelian group has a solvable word