Throughout this chapter, by a manifold, we shall mean a $C^\infty$ manifold, for simplicity of language. Vector fields, forms and other objects will also be assumed to be $C^\infty$ unless otherwise specified. We let $X$ be a manifold. We denote the $\mathbb{R}$-vector space of vector fields by $\Gamma T(X)$. Observe that $\Gamma T(X)$ is also a module over the ring of functions $\mathcal{F} = \mathcal{C}^\infty(X)$. We let

$$\pi: TX \to X$$

be the natural map of the tangent bundle onto $X$.

VIII, §1. BASIC PROPERTIES

By a covariant derivative $D$ we mean an $\mathbb{R}$-bilinear map

$$D: \Gamma T(X) \times \Gamma T(X) \to \Gamma T(X),$$

denoted by $(\xi, \eta) \mapsto D_\xi \eta$, satisfying the two conditions:

**COVD 1.** (a) In the first variable $\xi$, $D_\xi \eta$ is $\mathcal{F}$-linear.

(b) For a function $\varphi$, define $D_\xi \varphi = \xi \varphi = \mathcal{L}_\xi \varphi$ to be the Lie derivative of the function. Then in the second variable $\eta$, $D_\xi \eta$ is a derivation. Thus (a) and (b) can be written in the form:

$$D_{\varphi \xi} \eta = \varphi D_\xi \eta \quad \text{and} \quad D_{\xi} (\varphi \eta) = (D_{\xi} \varphi) \eta + \varphi D_{\xi} \eta.$$

**COVD 2.** $D_\xi \eta - D_\eta \xi = [\xi, \eta]$. 
Remark. This second condition can be eliminated to give rise to a more general notion, following the ideas of a connection as described at the end of Chapter IV, §3. However, we concentrate here on what we need for some basic results, rather than develop systematically the general theory of connections.

Having defined $D_\xi$ on functions and vector fields, we may extend the definition to all differential forms, or even to multilinear tensor fields. Let $\omega$ be in $\Gamma L'(T(X))$, i.e. $\omega$ is a multilinear tensor field on $X$, not necessarily alternating. We define $D_\xi\omega$ by giving its value on vector fields $\eta_1, \ldots, \eta_r$, namely

$$(D_\xi\omega)(\eta_1, \ldots, \eta_r) = \mathcal{L}_\xi(\omega(\eta_1, \ldots, \eta_r)) - \sum_{j=1}^r \omega(\eta_1, \ldots, D_\xi\eta_j, \ldots, \eta_r).$$

The definition of $D_\xi$ is such that $D_\xi$ satisfies the derivation property with respect to the $r + 1$ variables $\omega, \eta_1, \ldots, \eta_r$, that is

$$D_\xi(\omega(\eta_1, \ldots, \eta_r)) = (D_\xi\omega)(\eta_1, \ldots, \eta_r) + \sum_{j=1}^r \omega(\eta_1, \ldots, D_\xi\eta_j, \ldots, \eta_r).$$

Recall that $D_\xi = \mathcal{L}_\xi$ on function, as on the left side of this equation. Looking in a local chart shows that $D_\xi\omega$ is again a multilinear tensor field. It is immediate from the definition that if $\omega$ is alternating, then so is $D_\xi\omega$. In particular, $D_\xi$ is a derivation with respect to contractions and it is also a derivation with respect to the wedge product, that is:

**COVD 3.** $D_\xi(\omega \circ \eta_1) = (D_\xi\omega) \circ \eta_1 + \omega \circ D_\xi\eta_1$.

**COVD 4.** On the algebra of alternating forms, the covariant derivative $D_\xi$ is a derivation, in the sense that for two forms $\omega$ and $\gamma$, we have

$$D_\xi(\omega \wedge \gamma) = D_\xi\omega \wedge \gamma + \omega \wedge D_\xi\gamma.$$

The proof comes directly from the definition of the wedge product in Chapter V, §3. In the finite dimensional case, when a form is a sum of decomposable forms, i.e. wedge products of forms of degree 0 and 1, it follows that the above definition is the unique extension of $D_\xi$ to the algebra of differential forms. Furthermore, similarly to the formula of Proposition 5.1 of Chapter V, for the Lie derivative of a form, one has:

**COVD 5.** $(\mathcal{L}_\xi\omega)(\eta_1, \ldots, \eta_r) = (D_\xi\omega)(\eta_1, \ldots, \eta_r) + \sum_{i=1}^r \omega(\eta_1, \ldots, D_\xi\eta_i, \ldots, \eta_r)$, which is an alternative to

$$\mathcal{L}_\xi(\omega(\eta_1, \ldots, \eta_r)) = (D_\xi\omega)(\eta_1, \ldots, \eta_r) + \sum_{i=1}^r \omega(\eta_1, \ldots, D_\xi\eta_i, \ldots, \eta_r).$$

**COVD 6.** $(d\omega)(\xi_0, \xi_1, \ldots, \xi_r) = \sum_{i=0}^r (-1)^i(D_\xi\omega)(\xi_1, \ldots, \xi_{i-1}, \xi_0, \xi_{i+1}, \ldots, \xi_r)$. 