4.1 The Matrix Product: A Closer Look

Matrices are useful in studying systems of linear equations precisely because matrix multiplication is defined so that the coefficient matrix times the column of variables gives the column of constants for the system. We begin this chapter by investigating matrix multiplication from a more general point of view.

Products of Three Matrices

Suppose that a fruit dealer has a fleet of four large and six medium-sized vans that are used for deliveries. The van drivers have found that the most efficient way to pack a large van is to lay 12 large crates along the bottom and then stack 20 smaller crates on top. For a medium-sized van it is best to stack 8 large crates on the bottom and place 14 small crates on top. This fruit dealer always packs 40 pounds each of bananas and apples in his large crates (bananas are easily smashed and need strong crates) and 20 pounds each of apples and oranges in his small crates.

Suppose our problem is to find out how many pounds of apples, bananas, and oranges this fruit dealer can deliver each day. If we are counting
apples, we can count in two ways. As a first strategy we could figure out
the number of pounds of apples in each large van and each medium van,
multiply these numbers by the numbers of large and medium vans (in this
case 4 and 6), and add the result. Alternatively, we could figure out the
number of large crates and the number of small crates carried by the fleet of
vans, multiply these numbers by the number of pounds of apples each type
of crate carries, and add these numbers to find the result. Both methods are
valid strategies for counting the total pounds of apples carried.

Either of these strategies can be applied to bananas and oranges as well
as apples. Let us express both of these computations in terms of matrices.
Consider the matrices

\[
A = \begin{pmatrix} 40 & 20 \\ 40 & 0 \\ 0 & 20 \end{pmatrix}, \quad B = \begin{pmatrix} 12 & 8 \\ 20 & 14 \end{pmatrix}, \quad C = \begin{pmatrix} 4 \\ 6 \end{pmatrix}.
\]

In matrix \(A\) the first column entries are the pounds of apples, bananas, and
oranges in the large crates, and the second column gives those numbers for
the small crates. In matrix \(B\) the first column entries are the number of large
and small crates stacked in the large vans, and the second column gives
those numbers for the medium vans. Finally, matrix \(C\) is the column matrix
denoting the number of large and medium vans.

Suppose we are counting according to our first strategy. Then we will
compute the number of pounds of each fruit carried in each type of van. This
is given by the matrix product \(AB\), whose columns give this information:

\[
AB = \begin{pmatrix} 40 & 20 \\ 40 & 0 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} 12 & 8 \\ 20 & 14 \end{pmatrix} = \begin{pmatrix} 480 & 600 \\ 480 & 320 \\ 400 & 280 \end{pmatrix}.
\]

For example, we see that each large van carries 880 pounds of apples and
that the small vans each carry 600 pounds of apples. When we multiply this
product \(AB\) by the column matrix \(C\), we find the result of our first strategy,
namely

\[
(AB)C = \begin{pmatrix} 880 & 600 \\ 480 & 320 \\ 400 & 280 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 7120 \\ 3840 \\ 3280 \end{pmatrix}.
\]

In particular, we find a total of 7120 pounds of apples delivered daily.

Suppose we carried out the second strategy. In order to figure out the
number of large and small crates carried by the entire van fleet, we would
multiply the matrices \(B\) and \(C\), producing

\[
BC = \begin{pmatrix} 12 & 8 \\ 20 & 14 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 96 \\ 164 \end{pmatrix}.
\]

This means the fleet can carry a total of 96 large crates and 164 small crates.
Multiplying by our matrix \(A\) gives us the total number of pounds the fleet