CHAPTER

5

KEY CONCEPTS
OF LINEAR
ALGEBRA IN R^n

5.1 Linear Combinations and Subspaces

In this chapter we introduce the basic concepts of linear algebra. These ideas, namely linear independence, span, basis, and dimension, are crucial to all applications of this subject.

Equations for Studying a Network Flow Problem

In many problems where the methods of linear algebra are useful, it is necessary to analyze a large number of linear equations or conditions. In fact, at times so many equations arise that the computations look hopeless. Of course, one can use computers to study such problems, but again a large batch of equations can consume programming time. For this reason it becomes crucial to eliminate as many equations as possible that carry redundant information about the problem. The idea of redundancy of equations is made precise using the notion of a linear combination.
We illustrate the idea of linear combinations by considering a network flow problem and the various equations it gives. In Fig. 5.1 we have a graph with five vertices $A$, $B$, $C$, $D$, and $E$. We shall suppose that the graph represents possible freeway routes from point $A$ to point $E$ through a city. We shall assume that each segment in the graph has the carrying capacity indicated in cars per minute. Our problem is to determine the total carrying capacity between $A$ and $E$ of the freeway system (we will only consider one direction). We should also determine what the rate of flow should be along each freeway section in order to obtain the maximum capacity, which is important if we want to regulate traffic during peak hours.

In order to study this freeway network, we will work with seven variables, one to represent the traffic flow along each of the seven freeway segments. We let $X_{AB}$ denote the flow along the segment $AB$, $X_{AC}$ denote the flow along the segment $AC$, and so forth. Then, since the total flow, denoted $f$, is given by the total number of cars leaving $A$ and arriving at $E$, we have $f = X_{AB} + X_{AD}$ and $f = X_{BE} + X_{CE} + X_{DE}$. We write these first two equations as

$$X_{AB} + X_{AD} - f = 0,$$
$$X_{BE} + X_{CE} + X_{DE} - f = 0.$$  

(1)  

(2)

Also, since the flow into $B$ equals the flow out of $B$, and similarly for $C$ and $D$, we obtain

$$X_{BC} + X_{BE} - X_{AB} = 0,$$
$$X_{DC} + X_{DE} - X_{AD} = 0,$$
$$X_{BC} + X_{DC} - X_{CE} = 0.$$  

(3)  

(4)  

(5)

However, these are not all the possible equations. For example, we also have

$$X_{AB} + X_{DC} + X_{DE} - f = 0,$$  

(6)