8.1 Linear Transformations

Reflections in the Plane

We all learned at an early age that in spite of the similarities, there is a significant difference between a left shoe and a right shoe. How does a mathematician recognize and describe this difference? To a mathematician, the right shoe is the reflection of the left shoe, and vice versa. Look in the mirror at a left shoe’s reflection next to its companion right shoe to see why. It turns out that the difference between some objects and their reflections can be substantial, even more so than for shoes!

Consider the two pairs of triangles pictured in Fig. 8.1. They are reflections of one another across the $y$-axis. In spite of the fact that the reflections of the triangles are congruent, there is a fundamental difference between the two cases considered. The upper triangles, being isosceles, are also congruent through a rotation. In other words, the $180^\circ$ rotation centered at the point $(0, 3)$ moves one triangle onto the other. However, there is no way to
rotate the lower triangles onto one another. In fact, there is no combination of rotations or slides that can move one of the lower triangles onto the other. A reflection is needed to see their congruence (aside from cutting the triangles out of the page and flipping them over).

The difference between our lower triangles is the same as the difference between right and left shoes. A left shoe looks like a right shoe in the mirror, but unfortunately it will never fit well on a right foot. The reflection relationship is also important in organic chemistry, where it is known that reflections of chemical compounds can sometimes behave differently. If you imagine a molecule bouncing around in a gas or liquid you realize it can move in many directions and it can rotate in various ways, but it may not be able to transform into its reflection. In this chapter we study linear transformations, which are helpful in analyzing motions and symmetry. It is also important to understand how reflections and rotations are related, and for this we will need to use their matrix descriptions.

Matrix Descriptions of Reflections and Rotations

The reflection across the $y$-axis used in Fig. 8.1 is obtained algebraically by multiplying the $x$-coordinate of a point by $-1$. Hence it is given by left multiplication by the matrix

$$M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, if $\vec{v} \in \mathbb{R}^2$ is a column vector, then its reflection across the $y$-axis is the product $M_y \vec{v}$. We will denote this reflection function from $\mathbb{R}^2$ to $\mathbb{R}^2$ by the symbol $T_{M_y}$, and we write $T_{M_y} : \mathbb{R}^2 \to \mathbb{R}^2$ to demonstrate that the domain and range of the reflection function are $\mathbb{R}^2$. Our matrix representation shows for all $\vec{v} \in \mathbb{R}^2$ that $T_{M_y}(\vec{v}) = M_y \vec{v}$. 