MIRROR SYMMETRY AND ELLIPTIC CURVES

ROBBERT DIJKGRAAF

ABSTRACT. I review how recent results in quantum field theory confirm two general predictions of the mirror symmetry program in the special case of elliptic curves: (1) counting functions of holomorphic curves on a Calabi-Yau space (Gromov-Witten invariants) are 'quasi-modular forms' for the mirror family; (2) they can be computed by a summation over trivalent Feynman graphs.

1. INTRODUCTION

As discussed in detail by Kontsevich in this volume [Kon] the moduli space $M_g$ of algebraic curves has an interesting generalisation to the moduli space $M_g(X, d)$ of pairs $(C, f)$ with $C$ a genus $g$ curve and $f : C \to X$ a degree $d$ holomorphic map into a variety $X$. Tautological cohomology classes in the stable compactification $\overline{M}_g(X, d)$ are known as Gromov-Witten invariants. They appeared in Gromov's fundamental work on pseudo-holomorphic curves in symplectic geometry [Gro] and Witten's equally fundamental study of topological sigma models [Wit]. In the special case of genus zero curves, Gromov-Witten invariants are directly related to the quantum cohomology of the variety $X$ [LVW] and the symplectic Floer cohomology of the loop space $LX$ [Flo].

The moduli space $M_g(X, d)$ is also the primary object of study in the mirror symmetry program [Yau]. Mirror symmetry is concerned with counting the number of holomorphic curves on Calabi-Yau manifolds, i.e. compact Kähler manifolds $X$ with trivial canonical bundle $K_X$. One tries to define and calculate the generating functions

\[ F_g(t) = \sum_d N_{g,d} q^d, \quad q = e^{2\pi i t}, \]

where $N_{g,d}$ is the appropriately defined 'number' of genus $g$, degree $d$ curves on $X$. It can for example be given by the (orbifold) Euler character of $M_g(X, d)$.

In the above we assumed for convenience that $H^2(X)$ is one-dimensional and generated by the Kähler form $\omega$; otherwise, the degree is actually
a multi-degree and $F_g$ a multi-variable function. The above definition should also be slightly modified in the case $g = 0$ or 1, since these curves are not stable. For rational curves $C \cong \mathbb{P}^1$ we pick three hypersurfaces $H_0, H_1, H_\infty \subset X$, Poincaré dual to $\omega$, and consider maps $x : \mathbb{P}^1 \to X$ such that $x(z) \in H_z$ for $z = 0, 1, \infty$. This then defines the third derivative $F''_0$ of $F_0$. In case of an elliptic curve $C \cong E$ we pick a point $0 \in E$ and demand $x(0) \in H_0$, which then gives $F'_1$. In this note we will however be mainly concerned with the case $g > 1$.

The generating functions $F_g(t)$ are more or less by definition topological or, more precisely, symplectic manifold invariants of $X$. They do not depend on the complex structure of $X$, i.e. on the particular point in the moduli space $\mathcal{M}_X$ of manifolds of type $X$, but there is the obvious dependence on the parameter $t \in H^2(X)$, that labels the Kähler or symplectic class. The mirror conjecture states that for a Calabi-Yau manifold the functions $F_g(t)$ have an alternative interpretations as complex manifold invariants of a family of ‘mirror’ Calabi-Yau manifolds $\tilde{X}_t$, where $t$ is now interpreted as a suitable coordinate on $\mathcal{M}_{\tilde{X}}$, the moduli space of manifolds of type $\tilde{X}$.

Until recently most calculations were concerned with genus zero, where mirror symmetry is supposed to relate the function $F_0(t)$, that computes (part of) the quantum cohomology of $X$, to variation of Hodge structures for the family $\tilde{X}_t$. The precise formulation of the mirror symmetry conjecture for higher genus, i.e. the interpretation of the objects $F_g(t)$ in terms of the geometry of the mirror family $\tilde{X}_t$, was not clear. This has changed remarkably with the beautiful work of Bershadsky, Cecotti, Ooguri and Vafa [BCOV]. They have indicated the nature of the objects associated to $\tilde{X}$ that are conjecturally equivalent to the invariants $F_g$ associated to $X$, at least for the case of Calabi-Yau three-folds. This leads to two interesting predictions:

First, $F_g(t)$ should be a meromorphic object that can be obtained as the limit

$$F_g(t) = \lim_{t \to \infty} F_g^*(t, \bar{t})$$

of a non-holomorphic section $F_g^*$ of the line bundle $L^{g(2g-2)}$ over $\mathcal{M}_{\tilde{X}}$. Here $L$ is the bundle of holomorphic 3-forms with fiber $H^0(K_X)$. Sections of powers of this line bundle can be considered as generalizations of modular forms. The limiting holomorphic objects $F_g$ will have anomalous transformation properties, and will be named quasimodular forms. So, roughly we have:

**Claim 1** — The counting functions $F_g(t)$ of holomorphic curves on $X$ are quasimodular forms for the mirror family $\tilde{X}_t$.

Since under suitable circumstances the space of these quasimodular forms