5
Analysis in the 2-Cyclic Case

For the class of problems studied in this chapter, we succeed in obtaining quantitative convergence statements for the classical methods (Jacobi, Gauß-Seidel, and SOR iteration).

5.1 2-Cyclic Matrices

First, we define the term «weakly 2-cyclic» for matrices and for the pair \( \{ A, D \} \), where \( D \) is the diagonal part or block-diagonal part of \( A \).

**Definition 5.1.1.** A matrix \( A \in \mathbb{K}^{I \times I} \) is called *weakly 2-cyclic* (or *weakly cyclic of index 2*), if a block structure \( \{ I_1, I_2 \} \) with nonempty index subsets \( I_1, I_2 \subseteq I \) exists such that

\[
a_{\alpha \beta} = 0 \quad \text{for } \alpha, \beta \in I_1 \text{ as well as for } \alpha, \beta \in I_2. \quad (5.1.1)
\]

Condition (1) means that the diagonal-blocks are vanishing:

\[
A^{11} = 0, \quad A^{22} = 0. \quad (5.1.1')
\]

Often, not \( A \) but \( A - D \) has the form required in (1). In this case, we introduce the same name for the pair \( \{ A, D \} \).

**Definition 5.1.2.** The pair \( \{ A, D \} \), \( A, D \in \mathbb{K}^{I \times I} \), is called *weakly 2-cyclic*, if \( A - D \) is weakly 2-cyclic. An equivalent statement is that a block structure \( \{ I_1, I_2 \} \) with nonempty index subsets \( I_1, I_2 \subseteq I \) exists such that

\[
D = \text{blockdiag}\{A^{11}, A^{22}\}. \quad (5.1.2)
\]
Let $B$ be the block structure $\{I_1, I_2\}$ from Definition 1. Denote the block-diagonal part of a matrix (with respect to $B$) by $\text{blockdiag}_B(\cdot)$. Then $A$ is weakly 2-cyclic if and only if

$$\text{blockdiag}_B(A) = 0.$$ (5.1.1’)

The pair $\{A, D\}$ is weakly 2-cyclic if and only if

$$\text{blockdiag}_B(A) = D.$$ (5.1.2’)

The additional term «weakly» in front of «2-cyclic» indicates that the ordering of the indices is irrelevant. This is different in

**Definition 5.1.3.** $A$ [or $\{A, D\}$] is called 2-cyclic if the index set $I$ is ordered and the matrix $A$ [or the pair $\{A, D\}$] is weakly 2-cyclic with respect to the blocks $I_1 = \{1, \ldots, n_1\}$ and $I_2 = \{n_1 + 1, \ldots, n\}$ for a suitable $n_1$ with $1 \leq n_1 \leq n - 1$.

$1 \leq n_1 \leq n - 1$ ensures that both $I_1$ and $I_2$ are nonempty. The property «2-cyclic» is different from the property «cyclic of index 2» as, e.g., introduced by Varga [2, page 35]. A 2-cyclic matrix $A$ has the form

$$A = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \text{ } \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}.$$ (5.1.3a)

Note that, in general, $A_1 = A^{12} \in \mathbb{K}^{I_1 \times I_2}$ and $A_2 = A^{21} \in \mathbb{K}^{I_2 \times I_1}$ are not square block-matrices. The pair $\{A, D\}$ is 2-cyclic if

$$A = \begin{pmatrix} D_1 & A_1 \\ A_2 & D_2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad A - D = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}. \quad (5.1.3b)$$

The definitions immediately imply the following remark.

**Remark 5.1.4.**

(a) The property «2-cyclic» for a special ordering of the indices implies «weakly 2-cyclic» for any ordering of the indices.

(b) The property «weakly 2-cyclic» is independent of the ordering of the indices, whereas in the case of the term «2-cyclic» the indices may be permuted only inside of the respective blocks $I_1, I_2$.

(c) Let $A$ [or $\{A, D\}$] be weakly 2-cyclic. If $I$ is not ordered, then there is an ordering of the indices, so that $A$ [or $\{A, D\}$] is 2-cyclic with respect to this ordering. If $I$ is already ordered, there is a permutation of the indices with a corresponding permutation matrix $P$, so that $A := PAP^T$ [or $\{\hat{A}, \hat{D} := PDP^T\}$] is 2-cyclic.

Examples of (weakly) 2-cyclic matrices are given below for the case of the model problem.