5 Separation of variables and Fourier series

5.1 Two point boundary-value problems

In the applications which we will study in this chapter, we will be con­fronted with the following problem.

Problem: For which values of $A$ can we find nontrivial functions $y(x)$ which satisfy

$$\frac{d^2y}{dx^2} + Ay = 0; \quad ay(0) + by'(0) = 0, \quad cy(l) + dy'(l) = 0? \quad (1)$$

Equation (1) is called a boundary-value problem, since we prescribe infor­mation about the solution $y(x)$ and its derivative $y'(x)$ at two distinct points, $x = 0$ and $x = l$. In an initial-value problem, on the other hand, we prescribe the value of $y$ and its derivative at a single point $x = x_0$.

Our intuitive feeling, at this point, is that the boundary-value problern (1) has nontrivial solutions $y(x)$ only for certain exceptional values $\lambda$. To wit, $y(x) = 0$ is certainly one solution of (1), and the existence-uniqueness theorem for second-order linear equations would seem to imply that a solution $y(x)$ of $y'' + \lambda y = 0$ is determined uniquely once we prescribe two additional pieces of information. Let us test our intuition on the following simple, but extremely important example.

Example 1. For which values of $\lambda$ does the boundary-value problern

$$\frac{d^2y}{dx^2} + \lambda y = 0; \quad y(0) = 0, \quad y(l) = 0 \quad (2)$$

have nontrivial solutions?
### Solution.

(i) $\lambda = 0$. Every solution $y(x)$ of the differential equation $y'' = 0$ is of the form $y(x) = c_1 x + c_2$, for some choice of constants $c_1$ and $c_2$. The condition $y(0) = 0$ implies that $c_2 = 0$, and the condition $y(l) = 0$ then implies that $c_1 = 0$. Thus, $y(x) = 0$ is the only solution of the boundary-value problem (2), for $\lambda = 0$.

(ii) $\lambda < 0$: In this case, every solution $y(x)$ of $y'' + \lambda y = 0$ is of the form $y(x) = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$, for some choice of constants $c_1$ and $c_2$. The boundary conditions $y(0) = y(l) = 0$ imply that

$$c_1 = c_2 = 0, \quad e^{\sqrt{-\lambda} l} c_1 + e^{-\sqrt{-\lambda} l} c_2 = 0. \quad (3)$$

The system of equations (3) has a nonzero solution $c_1, c_2$ if, and only if,

$$\det\begin{pmatrix} 1 & 1 \\ e^{\sqrt{-\lambda} l} & e^{-\sqrt{-\lambda} l} \end{pmatrix} = e^{-\sqrt{-\lambda} l} - e^{\sqrt{-\lambda} l} = 0.$$ 

This implies that $e^{\sqrt{-\lambda} l} = e^{-\sqrt{-\lambda} l}$, or $e^{2\sqrt{-\lambda} l} = 1$. But this is impossible, since $e^z$ is greater than one for $z > 0$. Hence, $c_1 = c_2 = 0$ and the boundary-value problem (2) has no nontrivial solutions $y(x)$ when $\lambda$ is negative.

(iii) $\lambda > 0$: In this case, every solution $y(x)$ of $y'' + \lambda y = 0$ is of the form $y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$, for some choice of constants $c_1$ and $c_2$. The condition $y(0) = 0$ implies that $c_1 = 0$, and the condition $y(l) = 0$ then implies that $c_2 \sin \sqrt{\lambda} l = 0$. This equation is satisfied, for any choice of $c_2$, if $\sqrt{\lambda} l = n\pi$, or $\lambda = n^2 \pi^2 / l^2$, for some positive integer $n$. Hence, the boundary-value problem (2) has nontrivial solutions $y(x) = c \sin n\pi x / l$ for $\lambda = n^2 \pi^2 / l^2$, $n = 1, 2, \ldots$.

**Remark.** Our calculations for the case $\lambda < 0$ can be simplified if we write every solution $y(x)$ in the form $y = c_1 \cosh \sqrt{-\lambda} x + c_2 \sinh \sqrt{-\lambda} x$, where

$$\cosh \sqrt{-\lambda} x = \frac{e^{\sqrt{-\lambda} x} + e^{-\sqrt{-\lambda} x}}{2}$$

and

$$\sinh \sqrt{-\lambda} x = \frac{e^{\sqrt{-\lambda} x} - e^{-\sqrt{-\lambda} x}}{2}.$$ 

The condition $y(0) = 0$ implies that $c_1 = 0$, and the condition $y(l) = 0$ then implies that $c_2 \sinh \sqrt{-\lambda} l = 0$. But $\sinh z$ is positive for $z > 0$. Hence, $c_2 = 0$, and $y(x) = 0$.

Example 1 is indicative of the general boundary-value problem (1). Indeed, we have the following remarkable theorem which we state, but do not prove.

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