In one of his last papers, Lord Rayleigh, in 1917, computed approximate solutions up to order 6. He also showed by means of a numerical example that the relative error is not greater than $2.5 \cdot 10^{-6}$. We give here a rigorous existence proof for permanent gravitational waves of infinite depth which is also constructive.

Tullio Levi-Civita (1925)

I have tried to avoid long numerical computations, thereby following Riemann's postulate that proofs should be given through ideas and not voluminous computations.

David Hilbert, Report on Number Theory (1897)

In this chapter we study the existence of nontrivial water waves in a channel of finite depth. As shown in Figure 71.1 we find that, in addition to the trivial parallel flow, there occur nontrivial wave motions at certain critical velocities $c$. Such waves were studied during the nineteenth century by British hydrodynamicists such as Airy, Stokes, Kelvin, and Rayleigh. They solved the linearized problems and calculated nonlinear approximations up to order 6. No convergence proofs, however, were given. The linearized theory of Airy (1845) shows that for

$$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}$$  \hspace{1cm} (1)$$

nontrivial waves occur with:

- $c$ propagation velocity of the wave;
- $\lambda$ wave length;
- $h$ average channel depth;
- $g$ gravitational acceleration.
Relations between propagation velocity and wave length, of which formula (1) is an example, are called dispersion relations, and one of the main goals in the theory of permanent water waves is to obtain such dispersion relations for various situations.

For about 20 years Levi-Civita (1873–1941) worked on a rigorous solution for the complete nonlinear problem. In 1925, he found a very complicated existence proof for channels of infinite depth. Analogously to Section 71.6, he calculated the solutions as a power series with respect to a small parameter $s$. The main difficulty, which he overcame by performing voluminous computations, was to show the convergence of the formal solution by using a majorant method.

In our functional-analytic approach this difficulty is avoided, since existence and analyticity of solutions follows from the main theorem of analytic bifurcation theory (Theorem 8.A). In addition, we obtain uniqueness results which Levi-Civita did not have. Our approach shows that, by using the abstract methods of functional analysis, the proofs can be given through ideas following Hilbert's requirement cited at beginning of this chapter.

Independently of Levi-Civita, the Russian mathematician Nekrasov (1883–1957) also worked on this wave problem. In 1921, he gave the first existence proof for channels of infinite depth by using a nonlinear integral equation and a complicated majorant method. Using the method of Levi-Civita, Struik, in 1926, proved the existence of permanent gravitational waves in channels of finite depth. This proof was even more complicated than the proof of Levi-Civita for channels of infinite depth.

The functional-analytic approach of this section goes back to the author. The monograph by Zeidler (1968) contains detailed historical remarks as well as applications of this method to various classical wave problems. There, we presented, for the first time, a number of new existence proofs. The key is Theorem 8.A by the author. Further applications may be found in Zeidler (1971), (1972a, b), (1973), (1977, S), and Beyer and Zeidler (1979).

From a physical point of view our main result in this chapter is as follows. We are given:

$h$ average channel depth;  
$\rho_0$ constant density of the liquid;  
$\lambda$ wave length;  
$p_0$ constant barometric pressure on the surface of the liquid.

We find that in a neighborhood of the critical velocity $c$, given by (1), there