The geometers who have investigated the equations of equilibrium or motion of thin plates or surfaces have distinguished two kinds of forces, one produced by extension or contraction and the other by the bending of surfaces. ... It has seemed to me that these two kinds of forces could be reduced to a single one, which always ought to be called tension or pressure; this force acts upon each element of a section, chosen at will, not only in a flexible surface but also in a solid.

Augustin Louis Cauchy (1827)

The principal creator of three-dimensional elasticity is Cauchy (1789–1857). Mainly for use in three-dimensional hydrodynamics, Euler (1707–1783) had introduced general mappings of regions and had created the associated calculus of partial derivatives, chain rules, Jacobian determinants, etc; he also had formulated the general principles of linear and angular momentum and had shown how to apply them to fluids.

Cauchy mastered all this and turned it to use in elasticity. ... It is fair to say that much of the algebra of vectors, matrices, and tensors grew out of Cauchy's work on the strain, local rotation, and stress in elastic bodies.

Clifford Ambrose Truesdell (1983)

For a sufficiently small volume element, the general small change in the position of a deformable body can be represented as the sum of a translation, a rotation, and an extension or contraction in three orthogonal directions.

Hermann Helmholtz (1858)

The following principle, which goes back to Cauchy, is fundamental in stress analysis. If one imagines a volume element which is taken from an elastic body, then the outer forces, the inertial forces and the stress forces which act on the surface of the volume element, are in an equilibrium.

Erich Trefftz (1928)

If methods can be found, which admit a complete understanding of a given system, starting from a single atom and ending with the entire body, only then will a deeper knowledge in elasticity theory be gained.

Adolf Busemann and Otto Föppl (1928)
There are many reasons why nonlinear elasticity is not widely known in the scientific community:

(i) It is basically a new science whose mathematical structure is only now becoming clear.
(ii) Reliable expositions of the theory often take a couple of hundred pages to get to the heart of the matter.
(iii) Many expositions are written in a complicated indicial notation that boggles the eye and turns the stomach.

Stuart Antman (1984)

The goal of elasticity theory is the computation of deformations of elastic bodies and the corresponding stress forces. These deformations need not necessarily be small. Unfortunately, at present, there is no general nonlinear existence theory available. This makes the study of the literature quite difficult. Lacking this comprehensive general theory, a great number of models are used which are based on different approximation assumptions. These assumptions, however, are often not explicitly formulated and their foundation seems doubtful. Difficulties arise mainly from the fact that often there is no strict distinction between the different regions which correspond to the undeformed and deformed body.

We will try to present here an approach which might help the reader to understand the different models and approximation assumptions from a general and rigorous point of view. In what follows, we want to describe our general strategy, which is represented schematically in Figure 61.1.

Basic Equations and Typical Difficulties

In Section 61.3 we formulate the general basic equations of nonlinear elasticity theory. They consist of:

(a) time-dependent equations of motion for the deformed region of the elastic body; and
(b) constitutive laws which describe the connection between deformation and the crucial stress tensor \( \tau \).

The constitutive laws reflect the specific properties of the different elastic materials. The two main difficulties are the following:

(i) The stress tensor \( \tau \) refers to the unknown deformed region of the body.
(ii) The nonlinear constitutive laws are by no means uniquely determined by general physical principles.

In order to avoid difficulty (i) we use, in Section 61.5, the following crucial procedure. We replace the stress tensor \( \tau \) in the deformed region with the reduced stress tensor \( \sigma \) in the undeformed region which is sometimes also called the first Piola–Kirchhoff stress tensor. Other than in Section 61.3 we thereby obtain basic equations in the undeformed region of the body. The key observation is that \( \tau \) in the deformed region can be computed from \( \sigma \) in the