In real situations parameters such as a priori probabilities are rarely known exactly. In some problems a priori probabilities can be estimated with a high degree of confidence. In situations where this cannot be done the minimax criterion is often an acceptable alternative for determining the best receiver operation. The minimax decision function corresponds to the Bayes decision function for the a priori probabilities which makes the Bayes risk a maximum. The minimax criterion has the additional characteristic of being independent of the actual a priori probabilities.

Let us consider two methods of determining minimax decision functions for binary problems, both of which are directly extendable to multiple decision functions. We shall restrict ourselves initially to binary decisions and also restrict consideration to signal spaces of two elements, \( s_0 \) and \( s_1 \).

To start with, certain relationships between average risk and conditional risks will be of value. We have noted that the conditional risks
are linear functions of $P_{01}$ and $P_{10}$. We assume, as before, that $C_{01} > C_{00}$ and $C_{10} > C_{11}$. Then

$$
\begin{align*}
C_{00} & \leq r(s_0, \delta) \leq C_{01} \\
C_{11} & \leq r(s_1, \delta) \leq C_{10}
\end{align*}
$$

(5.1)

Also,

$$
R(\pi, \delta) = E(r(s, \delta)) = \pi_0 r(s_0, \delta) + \pi_1 r(s_1, \delta)
$$

or, equivalently,

$$
R(\pi, \delta) = r(s_0, \delta) + [r(s_1, \delta) - r(s_0, \delta)]\pi_1
$$

(5.2)

Hence $R(\pi, \delta)$, for any $\delta$, is a linear function of the a priori probabilities, as indicated in Fig. 5.1. It is immediate that for any $\delta$

$$
\max_{\pi} R(\pi, \delta) = \max_{\pi} r(s, \delta)
$$

(5.3)

Thus

$$
\min_{\delta} \max_{\pi} R(\pi, \delta) = \min_{\delta} \max_{\pi} r(s, \delta)
$$

(5.4)

We define the $\delta$ which satisfies Eq. (5.4) as the minimax decision function, which we designate as $\delta_m$.

The minimax theorem, one of the fundamental theorems of decision theory, states that under very general conditions [5.4] (much more general than binary decision problems)

$$
\min_{\delta} \max_{\pi} R(\pi, \delta) = \max_{\pi} \min_{\delta} R(\pi, \delta)
$$

(5.5)

So, from Eqs. (5.4) and (5.5),

$$
\min_{\delta} \max_{\pi} r(s, \delta) = \max_{\pi} \min_{\delta} R(\pi, \delta)
$$

(5.6)

We define the least favorable a priori distribution $\pi_L$, as that one which maximizes $R(\pi, \delta_B)$, where $\delta_B$ is the Bayes decision rule for $\pi$.

Thus, by (5.6), the minimax decision rule can be found by finding the Bayes decision function corresponding to the least favorable a priori distribution.

From (5.2) we note that $R(\pi, \delta)$ is independent of the a priori distribution if and only if the conditional risks are equal.

We can now prove the following theorem.

**Theorem 5.1** For the binary decision problem assume that there exists a $\delta_m$ such that

$$
r(s_0, \delta_m) = r(s_1, \delta_m)
$$

(5.7)