3.1 SOME BASIC MATHEMATICAL PROBLEMS OF MULTIUSER SHANNON THEORY

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At the present state of development of multiuser Shannon theory, the main interest is in single-letter characterizations of achievable rate regions (capacity regions) of various source (channel) networks, such as source coding with side information, multiple descriptions, and broadcast channels. The mathematical background of most such problems is very similar, namely, an entropy or image size characterization in the sense of [1].

1. Entropy Characterization Problem.

For a discrete memoryless multiple source with generic variables \((X, Y_1, \ldots, Y_k)\), find a single-letter characterization of the closure of the set of all \((k+1)\)-dimensional vectors of the form

\[
\left\{ \frac{1}{n} H(X^n | f(X^n)), \frac{1}{n} H(Y^n_1 | f(X^n)), \ldots, \frac{1}{n} H(Y^n_k | f(X^n)) \right\}.
\]

Here \(n = 1, 2, \ldots\) and \(f\) is any function defined on the \(n\)th Cartesian power of the range of \(X\).

2. Image Size Characterization Problem.

The \(\eta\)-image size \(g_W(A, \eta)\) of a set \(A \subseteq X^n\) over a discrete memoryless channel (DMC) \(\{W : X \rightarrow Y\}\) is the minimum cardinality of \(B \subseteq Y^n\) such that \(W^n(B | x) \geq \eta\) for each \(x \in A\). The problem is to find, for a distribution \(P\) on \(X\) and DMCs \(\{W_i : X \rightarrow Y_i\}\), \(i = 1, \ldots, k\), a single-letter characterization of the limit of the sets of all \((k + 1)\)-dimensional vectors.
\[
\left\{ \frac{1}{n} \log |A|, \frac{1}{n} \log g_{W_1}(A, \eta), \ldots, \frac{1}{n} \log g_{W_k}(A, \eta) \right\}.
\]

Here \( A \subseteq X^n \) is any set of \( P \)-typical sequences, and \( 0 < \eta < 1 \) is fixed (the result is independent of \( \eta \)).

Both problems are solved for \( k = 2 \) (cf. [1]) but not for \( k \geq 3 \). An interesting (unsolved) special case of Problem 2 for \( k = 3 \) is the following: consider sets \( A \subseteq X^n \times Y^n \times Z^n \) consisting of triples of sequences which are jointly typical with respect to a given distribution on \( X \times Y \times Z \). Let \( A_1, A_2, \) and \( A_3 \) be the projections of \( A \) on \( X^n, Y^n, \) and \( Z^n \), respectively. Characterize the vectors (for \( n \to \infty \)) of form

\[
\left\{ \frac{1}{n} \log |A|, \frac{1}{n} \log |A_1|, \frac{1}{n} \log |A_2|, \frac{1}{n} \log |A_3| \right\}
\]
or at least those without the first component.

3. Divergence-Characterization Problem.

The analogue of the entropy-characterization problem for Kullback-Leibler divergence is relevant for hypothesis testing problems with communication constraints (cf. [2]). In case \( k = 1 \), the problem is to characterize, for two double sources with generic variables \( (X, Y) \) and \( (X, \bar{Y}) \), the closure of the set of all two-dimensional vectors

\[
\left\{ \frac{1}{n} H(f(X^n)), \frac{1}{n} D(P_f(X^n) Y^n \parallel P_f(X^n) \bar{Y}^n) \right\}.
\]


This, up to now, less investigated problem area includes jammer problems, Wyner's wiretap channel (cf.[1], p.407), and so on. Entropy and image size characterization problems underly many problems of this kind, as well.