G4.1. Introduction

In this appendix we present an axiomatic derivation of the field equations of continuum thermodynamics. The central premiss of the derivation is that any law of physics applicable to a continuous body should also apply to any sufficiently nice subregion of the body. Here we express the first and second laws of thermodynamics in this way, postulate constitutive assumptions about the form of the thermodynamic variables involved in these expressions, and derive equivalent field equations.

In order not to obscure the central ideas, we treat a simplified system, with no mechanical interactions and no non-contact internal interactions, and we do not present proofs. The extension of the theory to include such effects and the neglected proofs may be found in the references listed below.

To a great extent we follow our work [1967], with refinements developed in later work done partially in collaboration with MARTINS, MIZEL and NOLL.

G4.2. Preliminary Definitions

We consider a given body \( \mathcal{B} \), which we identify as a subset of three-dimensional space \( \mathcal{E} \), requiring it to be regularly closed, compact and to have rectifiable boundary. Associated with \( \mathcal{B} \) is a family \( \mathcal{S} \) of subbodies, subsets of \( \mathcal{B} \) which share the geometrical properties of \( \mathcal{B} \). \( \mathcal{S} \) is subject to certain axioms which we need not elaborate upon here\(^1\), except to note that for each \( \mathcal{A}, \mathcal{C} \in \mathcal{S} \) we require also \( \mathcal{A} \cup \mathcal{C} \in \mathcal{S} \).

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\(^1\) The problem of constructing families \( \mathcal{S} \) appropriate to continuum mechanics, briefly discussed in the references, is not yet satisfactorily resolved.

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We define the exterior $\mathcal{A}^e$ of a set $\mathcal{A} \subset \mathcal{E}$ as

$$\mathcal{A}^e = \text{closure} \left( \text{interior} \left( \mathcal{E} \setminus \mathcal{A} \right) \right),$$

and call $\mathcal{A}$ and $\mathcal{C}$ separate if $\mathcal{A} \subset \mathcal{C}^e$. In particular, if $\mathcal{A}$ and $\mathcal{C}$ are subbodies or exteriors of subbodies, then $\mathcal{A}$ and $\mathcal{C}$ are separate if they intersect at most along their boundaries. In this case we call the (possibly empty) set

$$\mathcal{I} = \mathcal{A} \cap \mathcal{C}$$

the contact surface of $\mathcal{A}$ and $\mathcal{C}$.

We will frequently deal with pairs $(\mathcal{A}, \mathcal{C})$ such that:

(i) $\mathcal{A}$ is a subbody,

(ii) $\mathcal{C}$ is a subbody or the exterior of a subbody,

(iii) $\mathcal{A}$ and $\mathcal{C}$ are separate;

for convenience we write

$$\mathcal{M} = \{ (\mathcal{A}, \mathcal{C}) | (i)-(iii) \text{ are satisfied} \}.$$

If $\mathcal{S}_0 \subset \mathcal{S}$ we call a function $F: \mathcal{S}_0 \to \mathbb{R}$ additive (respectively, super-additive) if

$$F(\mathcal{A} \cup \mathcal{C}) = F(\mathcal{A}) + F(\mathcal{C}) \quad \text{(respectively, } \geq F(\mathcal{A}) + F(\mathcal{C}))$$

for all separate $\mathcal{A}, \mathcal{C}$ for which $\mathcal{A}, \mathcal{C}, \mathcal{A} \cup \mathcal{C} \in \mathcal{S}_0$. Such a function is volume-bounded if for some $k \in \mathbb{R}$ and all $\mathcal{A} \in \mathcal{S}_0$,

$$|F(\mathcal{A})| \leq k V(\mathcal{A}).$$

Here we introduce $V$ as volume measure in $\mathcal{E}$; $A$ shall denote area measure.

If $G: \mathcal{M} \to \mathbb{R}$, we call $G$ biadditive if both of the maps

$$\mathcal{A} \mapsto G(\mathcal{A}, \mathcal{C}), \quad \mathcal{C} \mapsto G(\mathcal{A}, \mathcal{C})$$

are additive where defined; and area-bounded if for some $k \in \mathbb{R}$,

$$G(\mathcal{A}, \mathcal{C}) \leq k A(\mathcal{A} \cap \mathcal{C})$$

whenever $\mathcal{A}$ and $\mathcal{C}$ are separate subbodies.

A function $F: \mathcal{S}_0 \to \mathbb{R}$ is said to have density $f: \mathcal{B} \to \mathbb{R}$ if for almost every $x \in \mathcal{B}$,

$$f(x) = \lim_{n \to \infty} \frac{F(\mathcal{A}_n)}{V(\mathcal{A}_n)}$$

is valid for any sufficiently regular sequence of subbodies of $\mathcal{S}_0$ shrinking to $\{x\}$. Let $F: \mathcal{S}_0 \to \mathbb{R}$ have the form

$$F(\mathcal{A}) = \int_{\partial \mathcal{A}} f \cdot n \, dA,$$