8 The Discovery of Irrational Numbers

The attempt to apply rational arithmetic to a problem in geometry resulted in the first crisis in the history of mathematics. The two relatively simple problems—the determination of the diagonal of a square and that of the circumference of a circle—revealed the existence of new mathematical beings for which no place could be found within the rational domain.

— Tobias Dantzig (1884–1956)

The discovery of these "holes" is attributed to Pythagoras, founder of the celebrated Greek school of mathematics and philosophy in the sixth century B.C. The life of Pythagoras is shrouded in mystery, and the little we know about him is more legend than fact. This is partially due to an absence of documents from his time, but also because the Pythagoreans formed a secret society, an order devoted to mysticism, whose members agreed upon strict codes of communal life. There is some doubt whether many of the contributions attributed to Pythagoras were indeed his own, but there is no question that his teaching has had an enormous influence on the subsequent history of mathematics, an influence which lasted for more than two thousand years. His name, of course, is immortally associated with the theorem relating the hypotenuse of a right triangle to its two sides, even though there is strong evidence that the theorem had already been known to the Babylonians and the Chinese at least a thousand years before him. The theorem says that in any right triangle, the square of the hypotenuse is equal to the sum of the squares of the two sides: $c^2 = a^2 + b^2$ (Fig. 8.1). The Pythagorean Theorem is probably the most well known, and certainly the most widely used theorem in all of mathematics, and it appears, directly or in disguise, in almost every branch of it.

Now among all right triangles there is one which is of special importance to our discussion: the right triangle which at the same time is also isosceles; that is, for which $a = b$. Since we are free to choose our unit of length at will, we may assign to each side a unit length ($a = b = 1$). The Pythagorean Theorem then says
that \( c^2 = 1^2 + 1^2 = 2 \), and the hypotenuse \( c \) therefore has a length equal to the square root of 2, written \( \sqrt{2} \). With the square root key on any calculator we can find its approximate value: 1.41421.

There exist various methods, or algorithms,\(^1\) to find the approximate value of \( \sqrt{2} \) to any desired accuracy, even though one can never find its “exact” value, since this would require an infinite number of digits (just as with \( \pi \)). But this fact does not prevent us from being able to locate the point corresponding to \( \sqrt{2} \) on the number line, and locate it exactly. We choose as our unit of length the segment from 0 to 1 (Fig. 8.2). At the end point of this segment we erect a perpendicular, also of unit length. We now place a compass at 0, open it until its other arm coincides with the end point of the perpendicular, and swing an arc. The point where this arc intercepts the number line is the desired point, since its distance from 0 is equal to \( \sqrt{2} \). Note that the entire construction uses only two instruments—a straightedge and a compass—in agreement with the Greek tradition which required that all geometric constructions be done with just these two tools. Our construction also shows that the desired point lies somewhere between the points 1 and 2 on the number line.

But what kind of a number is \( \sqrt{2} \)? The Greeks, of course, assumed that it is a rational number, since this was the only kind of number they knew. But one day some unknown member of the Pythagorean school—perhaps Pythagoras himself—made the startling discovery that \( \sqrt{2} \) is not commensurable with the unit—that is, the two numbers don’t have a common measure. Now this is the same as saying that \( \sqrt{2} \) cannot be written as a ratio of two integers; for if it could, then from the equation \( \sqrt{2} = \)

---

\(^1\) An algorithm is a “recipe” consisting of a finite number of steps which, when followed, will lead to the solution of a mathematical problem. An algorithm for \( \sqrt{2} \) would enable us to find its decimal expansion to any given (finite) number of decimal places. The word “algorithm” comes from Al Khowarizmi, an Arab mathematician living in the ninth century who was mainly responsible for bringing the Hindu system of numeration into widespread use in Europe.