CHAPTER 10
Sample Path Properties at Upcrossings

Our main concern in the previous chapter has been the numbers and locations of upcrossings of high levels, and the relations between the upcrossings of several adjacent levels. For instance, we know from Theorem 9.3.2 and relation (9.2.3) that for a standard normal process each upcrossing of the high level $u = u_*$ with a probability $p = \tau^*/\tau$ is accompanied by an upcrossing also of the level

$$u_{\ast \ast} = u - \frac{\log p}{u},$$

asymptotically independently of all other upcrossings of $u_*$ and $u_{\ast \ast}$.

We shall in this chapter show that the empirical distributions of the values of $\xi(t)$ after a $u$-upcrossing converge, and shall represent the limiting distributions as the distribution of a certain model process. By studying in more detail the behaviour of this model process, we will then attempt to throw some further light on the structure of the sample paths of $\xi(t)$ near an upcrossing of a high level $u$.

10.1. Marked Upcrossings

We assume that $\{\xi(t)\}$ is a stationary normal process on the entire real line with $E(\xi(t)) = 0$, $E(\xi^2(t)) = 1$ and covariance function $r(\tau)$ satisfying

$$r(\tau) = 1 - \frac{1}{2} \lambda_2 \tau^2 + o(\tau^2) \quad \text{as } \tau \to 0. \quad (10.1.1)$$

With a slightly more restrictive assumption,

$$-r''(\tau) = \lambda_2 + O(\log |\tau|)^{-\alpha} \quad \text{as } \tau \to 0 \quad (10.1.2)$$
for some \( a > 1 \), we can assume that \( \{\zeta(t)\} \) has continuously differentiable sample paths, (cf. condition (7.3.2) for sample function continuity) and we will do so since it serves our purposes of illustration. We also assume throughout this chapter that for each choice of distinct nonzero points \( s_1, \ldots, s_n \), the distribution of \( \zeta(0), \zeta'(0), \zeta(s_1), \ldots, \zeta(s_n) \) is nonsingular. (A sufficient condition for this is that the spectral distribution function \( F(\lambda) \) has a continuous component; see Cramér and Leadbetter (1967, Section 10.6).)

Since \( \lambda_2 = -r''(0) < \infty \) the number of upcrossings of the level \( u \) in any bounded interval has a finite expectation and so will be finite with probability one. Let

\[
0 < t_0 < t_1 < t_2 < \cdots
\]

with \( t_0 \leq 0 < t_1 \), be the locations of the upcrossings of \( u \) by \( \{\zeta(t)\} \), and note that \( |t_k| \to \infty \) as \( |k| \to \infty \). As before, we denote by \( N_u \) the point process of upcrossings with events at \( \{t_k\} \).

In order to retain information about \( \{\zeta(t)\} \) near its upcrossings, we now attach to each \( t_k \) a mark \( \eta_k \). In Chapter 7 each mark was simply a real number (e.g. in Section 7.6 where the marks were the values of \( \zeta(t) \) at the downcrossing zeros of \( \zeta'(t) \)). Here the useful marks are more abstract, and in fact we take \( \eta_k \) to be the function defined by

\[
\eta_k(t) = \zeta(t_k + t).
\]

Thus, the mark \( \{\eta_k(t)\} \) is the entire sample function of \( \{\zeta(t)\} \) translated back by the distance \( t_k \). By assumption \( \zeta \) is continuously differentiable and has finite number of upcrossings in finite intervals. Since further \( \zeta(t_k) > 0 \) at any upcrossing it is easily seen that each \( \eta_k(t) \) is a r.v. For, with \( t_k^{(n)} = \min\{i/n, i = 1, 2, \ldots; \zeta((i - 1)/n) < u < \zeta(i/n)\}, \) clearly \( t_k^{(n)} \to t_k \) and \( \zeta(t_k^{(n)} + t) \to \zeta(t_k + t) \), a.s., so that \( \eta_k(t) \), being a limit of r.v.'s is a r.v. In general, let \( t_k^{(n)} = \min\{i/n, i = 1, 2, \ldots; \zeta(t_k - 1 + (i - 1)/n) < u < \zeta(t_k - 1 + i/n)\}, \) leading to \( \zeta(t_k - 1 + t_k^{(n)} + t) \to \zeta(t_k + t) \), and hence also \( \eta_k(t) \) is a r.v.

In particular, \( \eta_k(0) = u \), while for small \( t \)-values \( \eta_k(t) \) describes the behaviour of the \( \zeta \)-process in the immediate vicinity of its \( k \)-th upcrossing

![Figure 10.1.1. Point process \( \{t_k\} \) of upcrossings for \( \zeta(t), t \geq 0 \).](image-url)