13.1 MORE INCIDENCE THEOREMS

Before thinking about adding any new axioms to our system, we shall prove several more incidence theorems that follow from the three axioms we already have. Among these is the very useful little theorem mentioned in Section 9.2 that is known as Crossbar. Crossbar can be stated as a theorem only after a formal definition of the interior of an angle has been given. We begin by extending the definition of in and on so that our theory encompasses such phrases as "ΔABC is on a side of line l" and "CD is in the ray-interior of ∠AVB." Also, for example, we shall be able to talk about two segments being on opposite sides of line l.

**DEFINITION 13.1** If S and T are nonempty sets of points and $S \subseteq T$, then S is on T or S is in T.

**Theorem 13.2** Let each of $A, B, C, D$ be either a point or a nonempty set of points. Let $A$ and $B$ be on opposite sides of line $l$. Then $A$ and $B$ are not on the same side of $l$. If $B$ and $C$ are on opposite sides of $l$, then $A$ and $C$ are on the same side of $l$. If $B$ and $D$ are on the same side of $l$, then $A$ and $D$ are on opposite sides of $l$.

**Proof** Trivial (Definition 12.9, Definition 13.1, and Theorem 12.10).
Theorem 13.3  If $S$ is a nonempty convex set which does not intersect line $l$, then $S$ is on one side of $l$.

Proof  Let $A$ be a point of $S$ on side $H_1$ of line $l$. Assume there exists a point $B$ of $S$ on the opposite side of $H_1$. Then $AB$ intersects $l$ by PSP. So, since $S$ is a convex set (Definition 8.18), we have the contradiction that $S$ intersects $l$. Hence all the points of $S$ are on $H_1$.  

Corollary 13.4  If $x$ is a line, ray, or segment which does not intersect line $l$, then $x$ is on one side of $l$. If line $l$ intersects $\overrightarrow{AC}$ only at point $V$ such that $A - V - C$, then $\text{int}(\overrightarrow{VA})$ and $\text{int}(\overrightarrow{VC})$ are on the same side of $l$ as is $A$ but $\text{int}(\overrightarrow{VA})$ and $\text{int}(\overrightarrow{VC})$ are on opposite sides of $l$.

Definition 13.5  The interior of $\angle AVB$ is the intersection of the side of $\overrightarrow{VA}$ that contains $B$ and the side of $\overrightarrow{VB}$ that contains $A$; $\text{int}(\angle AVB)$ is the interior of $\angle AVB$.

The interior of $\angle AVB$ is illustrated in Figure 13.1. That the interior of an angle is well-defined follows from Corollary 13.4. From the definition and the corollary preceding it, we have several immediate results that are lumped together as the next theorem.

Theorem 13.6  Point $P$ is in $\text{int}(\angle AVB)$ iff points $A$ and $P$ are on the same side of $\overrightarrow{VB}$ and points $B$ and $P$ are on the same side of $\overrightarrow{VA}$. Given $\angle AVB$, if $A - P - B$, then $P$ is in $\text{int}(\angle AVB)$. Given $\triangle ABC$, then $\text{int}(\overrightarrow{AB})$ is on $\text{int}(\angle ACB)$.

\begin{figure}[h]
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\includegraphics[width=0.5\textwidth]{figure13.1}
\caption{Figure 13.1}
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