CHAPTER 28

Rotations, Translations, and Horolations

28.1 PRODUCTS OF TWO REFLECTIONS

In this chapter we consider products of reflections in lines from one brush. In particular, such isometries fix those cycles that are concentric with the brush. Check Definition 20.1 if you don’t recall what a line of symmetry is.

**Theorem 28.1** If cycle \( \mathcal{A} \) and brush \( b \) are concentric, then \( l \) is a line of symmetry for \( \mathcal{A} \) iff \( l \) is in \( b \).

**Proof** If \( l \) is a line of symmetry for \( \mathcal{A} \), then \( l \) is the perpendicular bisector of a chord of \( \mathcal{A} \) and hence (Theorem 27.8) is in \( b \). Conversely, if \( l \) is in \( b \), then \( l \) is a line of symmetry for \( \mathcal{A} \) by the definition of \( \mathcal{A} \). 

Figure 28.1 should explain the statement of the next theorem. If \( b \) is a pencil with center \( C \), then every line in \( b \) intersects a circle with center \( C \) at two points. (Compare this with Theorems 27.11 and 27.12.) This explains the necessity of some additional hypothesis in the theorem when \( b \) is a pencil.

**Theorem 28.2** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two cycles concentric with brush \( b \). Let \( m \) and \( m' \) be two lines in \( b \) such that \( m \) intersects \( \mathcal{A} \) and \( \mathcal{B} \) at \( A \) and \( B \), respectively, and \( m' \) intersects \( \mathcal{A} \) and \( \mathcal{B} \) at \( A' \) and \( B' \), respec-
If $b$ is a pencil with center $C$, suppose $A-B-C$ and $A'-B'-C$. Then $AB=A'B'$ and the perpendicular bisector of $AA'$ is the perpendicular bisector of $BB'$.

**Proof** Let $l$ be the perpendicular bisector of $AA'$. So $\rho_l A = A'$, and $l$ is in $b$ (Theorem 27.8). Then, since $l$ and $m$ are in $b$ with $m = AB$, we have $\rho_m m$ is the line in $b$ through $A'$. Thus $\rho_m m = m'$ and $\rho_l B$ is on $m'$. Further, since $\rho_l \mathcal{B} = \mathcal{B}$ by the preceding theorem, we have $\rho_l B$ is on $\mathcal{B}$. Hence $\rho_l B = B'$. Therefore, $l$ is the perpendicular bisector of $BB'$ and $AB = A'B'$.

Complete familiarity with Theorem 19.10 will be assumed without further mention throughout our study of isometries. Let's recall the three important statements from this theorem: Every isometry is a product of at most three reflections. If an isometry fixes a point, then the isometry is either a reflection or a product of two reflections. If an isometry fixes two points on line $l$ then the isometry is either the reflection in $l$ or the identity.

If $a$ is a line, then $\rho_a \rho_a$ is the identity. We now consider $\rho_b \rho_a$ where $a$ and $b$ are distinct lines. Remember $\rho_b \rho_a$ is $\rho_a$ followed by $\rho_b$. 