Introduction

It is very important to study the monotonicity properties of distributions in order to obtain inequalities useful in statistical inference. Some monotonicity properties of distributions are well known and have proved to be very useful. During the last decade, more concepts have been introduced and used by several authors in multiple decision problems.

In Section 1.2., we discuss the distributions which have stochastically increasing property in a very general setting. The monotonicity of probability integrals of multivariate normal distribution is of special importance for the evaluation of multivariate normal probability integrals. We discuss it in Section 1.3. In Section 1.4, we study various types of monotonicity which are applicable without the assumption of normality. The relations between these various types are discussed. Some examples are given to show that none of the proposed definitions are equivalent and moreover to show the importance of all these concepts. The relation with some older and well-known concepts will also be considered. In Section 5, we discuss the monotonicity of distributions in terms Schur functions. Various useful densities will be given to show the importance of this idea. Applications to inequalities are given by Marshall and Olkin (1979) and Tong (1980).

1.2. Ordered Families of Distributions

Let $X = (X_1, \ldots, X_n)$ be a random vector with a probability distribution $P_\theta$, depending on a real parameter $\theta$. In most problems that one encounters in applications, such distributions are usually ordered, roughly speaking, in the sense that large values of $\theta$ lead, on the whole, to large values of the $X_i$'s. The concept of monotone likelihood ratio (MLR) due to Karlin and Rubin (1956) is very important in statistics. The concept of total positivity (see Karlin (1968)) is more general. In the case of total positivity of order 2 (TP2), if densities exist, then TP2 is equivalent to MLR. Further, MLR implies stochastically increasing property (SIP).

We now discuss several results relating to the stochastic ordering of distributions and the monotonicity of certain probability integrals.

1.2.1 Definition. A function $\varphi$ defined on $\mathbb{R}^m$, an m-dimensional Euclidean space, is said to be increasing for a partial order $\leq$, if $x_1 \leq x_2$ implies $\varphi(x_1) \leq \varphi(x_2)$.

1.2.2 Definition. A set $S$ in $\mathbb{R}^m$ is said to be increasing if its indicator set function is increasing, i.e., if $x_1 \in S$ and $x_1 \leq x_2$ then $x_2 \in S$.

Lehmann (1955) has shown that the following two conditions are equivalent:

(A) $P_\theta$ has SIP,
(B) If \( \varphi(x) \) is an increasing function, then \( E_{\theta P}(X) \) is increasing in \( \theta \).

Gupta and Panchapakesan (1972) give a more general result to provide sufficient conditions for the monotonicity problems as follows:

1.2.3 Theorem. Let \( \{F(\cdot | \lambda), \lambda \in \Lambda\} \) be a family of absolutely continuous distributions on the real line with continuous densities \( f(\cdot | \lambda) \) and \( \varphi(x, \lambda) \) a bounded real-valued function possessing first partial derivatives \( \varphi_x \) and \( \varphi_{\lambda} \) with respect to \( x \) and \( \lambda \) respectively and satisfying regularity conditions \((*)\). Then \( E_\lambda[\varphi(X, \lambda)] \) is nondecreasing in \( \lambda \) provided for all \( \lambda \in \Lambda \),

\[
(1.2.1) \quad f(x|\lambda) \frac{\partial \varphi(x, \lambda)}{\partial \lambda} - \frac{\partial F(x, \lambda)}{\partial \lambda} \frac{\partial \varphi(x, \lambda)}{\partial x} \geq 0 \quad \text{a.e.} \ x,
\]

where

\((*)\) \( (i) \) for all \( \lambda \in \Lambda \), \( \frac{\partial \varphi(x, \lambda)}{\partial x} \) is Lebesgue integrable on \( \mathbb{R} \); and

\( (ii) \) for every \([\lambda_1, \lambda_2] \subset \Lambda \) and \( \lambda_3 \in \Lambda \) there exists \( h(x) \) depending only on \( \lambda_i \), \( i = 1, 2, 3 \) such that

\[
\left| \frac{\partial \varphi(x, \lambda)}{\partial \lambda} f(x|\lambda_2) - \frac{\partial F(x|\lambda_3)}{\partial \lambda} \frac{\partial \varphi(x, \lambda_3)}{\partial x} \right| \leq h(x)
\]

for all \( \lambda \in [\lambda_1, \lambda_2] \) and \( h(x) \) is Lebesgue integrable on \( \mathbb{R} \).

Proof. Let us consider \( \lambda_1, \lambda_2 \in \Lambda \) such that \( \lambda_1 \leq \lambda_2 \) and define

\[
(1.2.2) \quad A_i(\lambda_1, \lambda_2) = \int \ldots \int_{\mathbb{R}} \frac{2}{r_1} \varphi(x, \lambda) dF_i(x), \quad i = 1, 2
\]

and

\[
(1.2.3) \quad B(\lambda_1, \lambda_2) = \sum_{i=1}^{\lambda_1} A_i(\lambda_1, \lambda_2),
\]

where \( F_i \equiv F_i \), \( i = 1, 2 \). We note that when \( \lambda_1 = \lambda_2 = \lambda \), \( B(\lambda, \lambda) = 2E_\lambda[\varphi(X, \lambda)] \).

Integrating \( A_i(\lambda_1, \lambda_2) \) by parts and using it in \((1.2.3)\), it is easily seen that

\[
(1.2.4) \quad B(\lambda_1, \lambda_2) = \text{a term independent of } \lambda_1
\]

\[
+ \int \varphi(x, \lambda_1) f_2(x) - F_1(x) \varphi(x, \lambda_2) dx.
\]

Hence,

\[
(1.2.5) \quad \frac{\partial}{\partial \lambda_1} B(\lambda_1, \lambda_2) = \int \varphi_{\lambda_1}(x, \lambda_1) f_2(x) - \frac{\partial}{\partial \lambda_1} F_1(x) \varphi(x, \lambda_2) dx
\]

and this is nonnegative if, for \( \lambda_1 \leq \lambda_2 \),

\[
(1.2.6) \quad \varphi_{\lambda_1}(x, \lambda_1) f_2(x) - \frac{\partial}{\partial \lambda_1} F_1(x) \varphi(x, \lambda_2) \geq 0
\]

for all \( x \).

Now, we consider the configuration \( \lambda_1 = \lambda_2 = \lambda \). It can be easily verified that