CHAPTER IV

Complete Systems of Commuting Observables

In this chapter it is explained that the question of what constitutes a complete system of commuting observables is not a mathematical question but can only be answered by experiment.

For the algebra of the quantum-mechanical harmonic oscillator, the eigenvectors of the operator $N$ [or of $H = \hbar \omega (N + \frac{1}{2})$],

$$N |n\rangle = n |n\rangle \quad (n = 0, 1, 2, \ldots) \quad (1.1)$$

constituted a complete orthonormal system of $\mathcal{H}$ [see (II.3.25)]. For the algebra of the quantum-mechanical rotator a complete orthonormal system in $\mathcal{H}$ was given by the eigenvectors of $J^2$ and $J_3$:

$$J^2 |jj_3\rangle = j(j + 1)\hbar^2 |jj_3\rangle \quad J_3 |jj_3\rangle = j_3\hbar |jj_3\rangle. \quad (1.2)$$

(We shall use either $j_3$ or $m$ to denote the quantum number labeling eigenvalues of $J_3$, depending on typographical convenience for the topic being discussed. In this chapter we use $j_3$.) Instead of the eigenvector $|n\rangle$ one can use the generalized eigenvectors $|x\rangle$ or $|p\rangle$ as a generalized basis for $\mathcal{H}$, as expressed by Equations (II.8.16') and (II.8.34). (Generalized basis vector of $\mathcal{H}$ also exist for the algebra of the quantum-mechanical rotator.)

The distinction between (1.1) and (1.2) is that for the one-dimensional oscillator (1.1), one operator is sufficient to define the basis. If for any vector $\phi \in \mathcal{H}$ it follows that

$$N \phi = a \phi, \quad (1.3)$$
then $a$ is one of the nonnegative integers, say $a = n'$, and $\phi = \alpha |n'\rangle$ where $\alpha \in \mathbb{C}$. For the rotator model, two operators are needed in general to define the basis, and two are sufficient. If for any vector $\psi \in \mathcal{H}$ it follows that

$$J^2 \psi = a \psi \quad \text{and} \quad J_3 \psi = b \psi,$$

then (1) $a = j'(j' + 1) \hbar^2$ and $b = j_3 \hbar$, where $j'$ is an integer or half integer, and $j_3$ is one of the numbers $-j', -j' + 1, \ldots, j'$; and (2) $\psi = \alpha |j'j_3\rangle$, where $\alpha \in \mathbb{C}$. Instead of $J^2$ and $J_3$ one could of course use any two independent (algebraic) functions $f_1(J^2, J_3)$ and $f_2(J^2, J_3)$ of $J^2$ and $J_3$, i.e., functions $f_i$ such that $j$ and $j_3$ are uniquely determined by the numbers $f_i(j, j_3)$ and vice versa. One would then obtain the same basis system.

Instead of the basis system (1.2) one could also use another basis system, e.g.,

$$J^2 |jj_2\rangle = j(j + 1) \hbar^2 |jj_2\rangle \quad J_2 |jj_2\rangle = j_3 \hbar |jj_2\rangle,$$  

or any two other functions of the operators $J_i$ ($i = 1, 2, 3$) that commute with each other. To obtain a (generalized) basis system of $\mathcal{H}$ one need not restrict oneself to functions of $J_i$, but could even use two functions of the $J_i, Q_i$ ($i = 1, 2, 3$) that commute with each other (e.g. $Q_i$ or $P_i$).* The eigenvalues of these functions usually specify a basis system of $\mathcal{H}$ completely.

[Sometimes, but only in the case that at least one of these functions has a continuous spectrum, it may happen that in addition to the two (generalized) eigenvalues of these functions a further label is needed to specify the generalized basis system completely.]

The system of commuting Hermitian operators that specifies the (generalized) basis system completely is called (following Dirac) a complete system of commuting observables (c.s.c.o.). The (generalized) eigenvalues of a c.s.c.o. are called quantum numbers.

For the quantum-mechanical one-dimensional harmonic oscillator, the c.s.c.o. consists of the one operator $N$ (or $H$), but the operator $Q$ may also serve as the c.s.c.o. For the rotator the c.s.c.o. consists of two operators; $J^2$ and $J_3$ are a convenient choice.

From what was said above, and the fact that in the direct-product space one basis system is obtained as the direct product of the basis vectors in the two factor spaces, it is clear that one c.s.c.o. of the combination of two systems is given by the combination of the two c.s.c.o.’s for each subsystem.

For example, a c.s.c.o. for the vibrating rotator is given by

$$N, \quad J^2, \quad J_3.$$

For different algebras there are different c.s.c.o.’s. The larger the algebra (i.e., the more complicated the physical system), the larger is the number of operators in a c.s.c.o.

* Such a basis system is given in (3.76) of the Appendix to Section V.3.