CHAPTER 3
LINEAR ORDINARY DIFFERENTIAL EQUATIONS

3.0. Introduction.

We will often write a derivative $d/dx$, as $du$ and also $d^k u = u^{(k)}$. Thus a single $n$th order linear inhomogeneous ordinary differential equation may be written as

\begin{equation}
\label{3.0.1}
 u^{(n)} + a_1 u^{(n-1)} + \cdots + a_n u = g(x)
\end{equation}

where the coefficients $a_i$ depend on $x$. (3.0.1) may also be written as a system of first order ordinary differential equation by defining new dependent variables

\[ w_1 = u^{(n-1)}, \quad w_2 = u^{(n-2)}, \ldots, w_n = u. \]

Then since

\begin{equation}
\label{3.0.2}
 w_{k-1} = \frac{dw_k}{dx}, \quad k = 2, \ldots, n
\end{equation}

we write (3.0.1) as

\begin{equation}
\label{3.0.3}
 \frac{d}{dx} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} + \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ -1 & 0 & \cdots & 0 \\ \vdots & -1 & \cdots & 0 \\ 0 & \cdots & -1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\end{equation}

or symbolically

\[ \frac{d}{dx} \mathbf{w} + A\mathbf{w} = \mathbf{g}. \]

Conversely we can consider a system of $n$ first equations
Exercise 58. Demonstrate that by rational operations and differentiation any component of $\mathbf{w}$ in (3.0.4) satisfies an ordinary differential equation of at most $n$th order.

It is clear that we may treat systems of ordinary differential equations of any order either as a system of first order ordinary differential equations or as a single ordinary differential equation of some high order. Having demonstrated this equivalence it must also be mentioned that (3.0.1) and (3.0.4) each have some aspects that recommend study of it. From the point of view of clarity and theory it is more advantageous to consider the system (3.0.4). Even in describing the construction of a solution, (3.0.4) is found to be simpler to deal with. On the other hand, in terms of actual labor (3.0.1) is invariably simpler to solve. Especially since certain "tricks" for solving (3.0.1) have no counterpart for the system, (3.0.4). Moreover, unless care is taken (3.0.1) and (3.0.4) each produce spurious solutions when applied to the other. Therefore, in spite of the loss efficiency we will consider both the single equation and the system.

In keeping with the spirit of the earlier chapters, we shall be more interested in constructing approximate solutions than in the demonstration of the existence and uniqueness of solutions. We, therefore, state without proof: (see e.g., Coddington and Levinson, "Theory of Ordinary Differential Equations" [10]).

Theorem 301. For $A(z), f(z)$ analytic in the complex variable $z$ in a region $R$, the equation

(3.0.5) \[
\frac{d\mathbf{w}}{dz} = A(z)\mathbf{w} + f(z)
\]

possesses a unique analytic solution for $z \in R$, taking on the prescribed data