CHAPTER 1:
RANDOMNESS

1.1 Fundamentals.

The concept of randomness is fundamental in probability theory and statistics, but also most controversial. Among the many interpretations of terms like probability, likelihood, etc., we shall consider two in this course: the usual frequency approach (in this chapter) and the Bayesian one (in chapter 6: "Decision problems").

One should actually not speak of a single frequency approach, since there are several variations of it. That most commonly adopted in the textbook literature is to start from the idea of a random experiment and carry out the mathematical formalization as follows.

Starting from a sample space $X$ that may be completely unstructured, one views the outcome of the random experiment $E$ as a realization of a stochastic variable $x$ described by a probability measure $P$ given on $X$. The pure mathematician is wont to phrase this as follows: the value of the probability $P(S)$ should be defined for any subset $S \subseteq X$ belonging to a well-defined $\sigma$-algebra of subsets of the sample space. (We shall not, however, go into the measure-theoretical aspects in this course.) This is the mathematical model: the transition to phenomena of the real world is effected through a heuristic principle: the frequency interpretation of probabilities. If the experiment $E$ is repeated $n$ times independently and under equal conditions, then the relative frequency $f/n$ should be close to $P(S)$, where $f$ is the absolute frequency (number of times we get a value $x \in S$), if the sample size, $n$, is large enough.

While the idea behind this ancient principle is quite appealing, the above formulation is not quite clear on three points:

a) what is meant by "independently"?

b) how should one interpret the phrase "under equal conditions"?

c) how large is "large enough"?

It has often been argued that this sort of vagueness must always be expected when any mathematical model is interpreted in terms of the physical world. For example, when we use Euclidean plane geometry to describe and analyze measurements...
of length, angles and areas, we meet the same sort of difficulty when trying to relate notions like points, lines and areas to physical data. We continue to use the model only as long as no logical inconsistency is found or no serious discrepancy between model and data has been established empirically. This pragmatic attitude has been widely accepted, but doubts have been voiced by critics who claim that a more profound analysis is possible. To understand better how this can be done we shall take a look at the manner in which we actually use the model.

Simplifying drastically, we could say that from the above point of view probability and mathematical statistics are the study of bounded measures. While such a statement would undoubtedly describe much research activity in these fields quite accurately, it is a superficial point of view and of little help when we want to discuss the relation between theory and its application.

Randomness enters on three different levels. The first can be exemplified by a set of measurements of some physical constant like the speed of light. Here we would think of the variation encountered among the data as caused by imperfections in the experimental arrangement, imperfections which could, at least in principle, be eliminated or reduced by building better equipment or using a more efficient design for the experiment. We describe this variation in probabilistic terms, but probability plays here only a marginal role. On the second level randomness plays a more fundamental role. Assume that we measure the toxicity of a drug and use guinea pigs in the experiment. We would have to consider the apparent randomness caused by the biological variation in the population of animals used. We would always expect such variation, although its form and extent might vary between different populations. In a well-designed experiment we would like to be able to make precise statements about this variation; to eliminate it entirely does not seem possible. Most of experimental statistics falls into this category. To illustrate the third level, let us think of a Monte Carlo experiment in which we try to find the properties of a statistical technique by applying it to artificial data and studying the result, or in which a probabilistic limit theorem is examined by simulating it for large samples. Here we need randomness, and try to generate data in some way so that they have properties we would expect from random sequences. Another example bringing out this feature perhaps even more clearly occurs in the design of experiments when we inject randomness.