§ 14. The countable chain condition

The algebraic behavior of the regular open algebra of a topological space $X$ reflects, at least in part, the topological properties of $X$. One particular topological property of $X$, namely the possession of a countable base, has important algebraic repercussions, which we now proceed to study.

A Boolean algebra $A$ is said to satisfy the countable chain condition if every disjoint set of non-zero elements of $A$ is countable. (Two elements $p$ and $q$ of a Boolean algebra are disjoint if $p \land q = 0$; a set $E$ is disjoint if every two distinct elements of $E$ are disjoint.) The regular open algebra of a space with a countable base does satisfy the countable chain condition. Proof: select a countable base, and, given a disjoint class of non-empty regular open sets, find in each one a set of the base. An algebra satisfying the countable chain condition is sometimes called countably decomposable.

**Lemma 1.** A Boolean algebra $A$ satisfies the countable chain condition if and only if every set $E$ in $A$ has a countable subset $D$ such that $D$ and $E$ have the same set of upper bounds.

Proof. Assume first that the condition is satisfied and suppose that $E$ is a disjoint set of non-zero elements of $A$. Let $D$ be a countable subset of $E$ with the same set of upper bounds. If $E$ had an element not in $D$, the complement of such an element would be an upper bound of $D$ but not of $E$. Conclusion: $E = D$, and therefore $E$ is countable.
To prove the converse, assume now that the countable chain condition is satisfied and let $E$ be an arbitrary subset of $A$. Let $M$ be the ideal generated by $E$; the elements of $M$ are just those elements of $A$ that can be dominated by the supremum of some finite subset of $E$. It follows that $M$ and $E$ have the same set of upper bounds. Apply Zorn's lemma to find a maximal disjoint set, say $F$, of non-zero elements of $M$. Reasoning as in the preceding paragraph, we infer that $F$ and $M$ have the same set of upper bounds. Since the countable chain condition holds, the set $F$ is countable. Since each of the countably many elements of $F$ is dominated by the supremum of some finite subset of $E$, the union, say $D$, of all these finite sets is a countable subset of $E$ with the same set of upper bounds.

**COROLLARY.** A Boolean algebra that satisfies the countable chain condition is complete.

**Proof.** Every countable supremum is formable by definition; by Lemma 1 every conceivable supremum coincides with some countable one.

The countable chain condition got its name from its close relation to a condition in which ascending chains do explicitly occur. An ascending well-ordered chain in a Boolean algebra $A$ is a function that associates with each element $a$ of some well-ordered set an element $p_a$ of $A$ so that $p_a \leq p_\beta$ whenever $a \leq \beta$. The chain is strictly ascending if $p_a \neq p_\beta$ whenever $a < \beta$, and the chain is called countable in case the set of indices is countable.

**LEMMA 2.** If a Boolean algebra $A$ satisfies the countable chain condition, then every strictly ascending well-ordered chain in $A$ is countable.