Completion

(7) Prove that the minimal completion of an atomic algebra is isomorphic to the field of all subsets of the set of atoms.

(8) Prove that the minimal completion of a non-atomic algebra is non-atomic.

(9) Does the minimal completion of an algebra satisfying the countable chain condition satisfy that condition also?

§ 22. Boolean σ-spaces

A Baire set in a Boolean space is a set belonging to the σ-field generated by the class of all clopen sets. Clearly every Baire set in a Boolean space is a Borel set; the converse is not true in general. A trivial way to manufacture open Baire sets is to form the union of a countable class of clopen sets. The converse is true but not trivial. The converse implies that every open Baire set is an $F_\sigma$ (that is, the union of a countable class of closed sets), and, consequently, every closed Baire set is a $G_δ$ (that is, the intersection of a countable class of open sets). We shall prove the main result about the structure of open Baire sets by proving first that every closed Baire set is a $G_δ$. Observe that in a metric space every closed set is a $G_δ$; in a general topological space this not so. The proof of the following auxiliary result uses the fact about metric spaces just mentioned; the trick is to construct a suitable metric space associated with each given closed Baire set.

**Lemma 1.** Every closed Baire set is a $G_δ$.

**Proof.** Let $F$ be a closed Baire set in the Boolean space $X$, and let $\{P_n\}$ be a sequence of clopen sets such that $F$
belongs to the \( \sigma \)-field generated by \( \{P_n\} \) (see Exercise 13.7). Let \( p_n \) be the characteristic function of \( P_n \) and write

\[
  d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |p_n(x) - p_n(y)|
\]

for all \( x \) and \( y \) in \( X \). The function \( d \) is a metric except perhaps for strict positiveness. If, in other words, two points \( x \) and \( y \) are defined to be equivalent, \( x \equiv y \), in case \( d(x, y) = 0 \), then the equivalence classes may be more than singletons. (It is trivial that the relation so defined is an equivalence.) Let \( U \) be the set of all equivalence classes. There is a natural mapping \( T \) from \( X \) onto \( U \); the value of \( T(x) \) is the equivalence class of \( x \), for each \( x \) in \( X \). If \( T(x_1) = T(x_2) \) and \( T(y_1) = T(y_2) \), then

\[
  d(x_1, y_1) \leq d(x_1, x_2) + d(x_2, y_2) + d(y_2, y_1) = d(x_2, y_2),
\]

and, by symmetry, the reverse inequality is also true, so that \( d(x_1, y_1) = d(x_2, y_2) \). This implies that writing

\[
  e(u, v) = d(x, y),
\]

whenever \( u = T(x) \) and \( v = T(y) \), unambiguously defines a metric \( e \) on \( U \). The inverse image (under \( T \)) of each open sphere in \( U \) is an open set in \( X \), so that \( T \) is continuous. A set in \( X \) is the inverse image of some set in \( U \) if and only if it consists of (that is, is the union of) equivalence classes. The class of sets with this property is a \( \sigma \)-field. If \( x \equiv y \), that is \( d(x, y) = 0 \), then \( p_n(x) = p_n(y) \) for all \( n \), so that \( x \) and \( y \) belong to the same \( P_n \)'s; this implies that each \( P_n \) belongs to the \( \sigma \)-field of unions of equivalence classes. It follows from the definition of generated \( \sigma \)-field that \( F \) also belongs to that \( \sigma \)-field, and hence that \( F = T^{-1}(V) \) for some subset \( V \) of \( U \). Since \( T(T^{-1}(V)) = V \), we infer that \( V \)