3.1 Jacobians

Let \( P \) be a rectangular parallelopiped in \( \mathbb{R}^k \), with edges along the Cartesian coordinate axes. We call such a figure a Cartesian rectangle. The figure shows a Cartesian rectangle in \( \mathbb{R}^2 \). Let the edges of \( P \) have lengths \( dx_1, dx_2, \ldots, dx_k \). Then its \( k \)-dimensional volume, denoted \( \text{Vol}(P) \), is by definition

\[
\text{Vol}(P) = dx_1 \, dx_2 \ldots \, dx_k,
\]

the product of its edge lengths. Let \( T \) be a \( k \)-by-\( k \) matrix. We may apply \( T \) to each vector lying in \( P \) and in this way transform \( P \) to another figure \( T(P) \), which typically will no longer be a Cartesian rectangle, nor even rectangular. To calculate the volume \( \text{Vol}(T(P)) \) of \( T(P) \) we use the theorem on the geometric meaning of the determinant which states, in the present notation, that
\[
\frac{\text{Vol}(T(P))}{\text{Vol}(P)} = |T|, \tag{2}
\]

where \( |T| \) is the standard notation for the determinant of \( T \). \(^{(1)}\)

Putting this together with (1) we have

\[
\text{Vol}(T(P)) = |T| dx_1 dx_2 \ldots dx_k. \tag{3}
\]

Let a differentiable function \( \hat{f}: \mathbb{R}^k \rightarrow \mathbb{R}^k \) be given, and take for \( T \) the differential \( d\hat{f}(\hat{x}_o) \) of \( \hat{f} \) at some fixed point \( \hat{x}_o \), \( \hat{f} \) being interpreted as a transformation. Then by (3) we have

\[
\text{Vol}(d\hat{f}(\hat{x}_o)(P)) = |d\hat{f}(\hat{x}_o)| dx_1 dx_2 \ldots dx_k. \tag{4}
\]

If \( P \) is a sufficiently small rectangle then, by the Taylor development 1.3(12), for each point \( \hat{x} \) of \( P \) the image \( \hat{f}(\hat{x}) \) is well approximated by \( d\hat{f}(\hat{x}_o) \cdot \hat{x} \), where \( \hat{x}_o \) is, say, the midpoint of \( P \).

Therefore we would expect that, if \( P \) is sufficiently small, the volume \( \text{Vol} \hat{f}(P) \) of the image of \( P \) under \( \hat{f} \) should be well approximated by \( \text{Vol} d\hat{f}(\hat{x}_o)(P) \), in some appropriate sense which we shall not inquire into here. Assuming this reasonable expectation to be so, we may write, for \( P \) sufficiently small,

\[
\text{Vol} \hat{f}(P) \sim |d\hat{f}(\hat{x}_o)| dx_1 dx_2 \ldots dx_k. \tag{5}
\]

Assume now that \( \hat{f} \) is invertible on a set \( E \) containing \( P \), let \( \hat{g} \)

\(^{(1)}\) See Courant [1], volume II, p. 33 for the theorem in dimensions 2 and 3, and Schreier-Sperner [5], p. 63 for the general case.