A large class of deterministic optimal control problems are special cases of the stochastic optimal control problems considered previously. This is true both with respect to the construction of schemes as well as the proofs of convergence. In fact, the convergence proofs become much simpler in the deterministic setting.

In the present chapter and the next we will consider deterministic optimal control problems which are not special cases. We have two goals in mind. The first is to show the flexibility of the Markov chain approximation methods with regard to weakening assumptions. The second is to discuss several classes of problems that are of current interest but not covered by the results given so far. To contain the development somewhat, we will focus on the construction of numerical schemes and proofs of convergence for problems from the calculus of variations. Such problems arise in a wide variety of settings. Well known examples are classical mechanics and geometric optics (e.g., [28, 62]). A more recent example is the theory of large deviations of stochastic processes [59]. In many ways these problems are simpler than most of the problems treated previously. There are, however, some interesting new features that must be dealt with. For example, when we rewrite a calculus of variations problem as a control problem, the space of controls is usually unbounded. In this case, the notion of local consistency must be extended so that the “errors” in the approximation are properly bounded as a function of the control. The tightness of controls (in the relaxed control framework) is no longer automatic, and must be shown to follow from the formulation of the optimization problem (at least for any sequence of nearly optimal controls). These issues are considered in detail...
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in Section 14.2.

In many problems we must also consider costs that are discontinuous. If the discontinuity is in the stopping or exit cost, then convergence can be proved under a mild "controllability" condition. Discontinuities of this sort are considered in Section 14.2. The problem becomes much more difficult if the discontinuity is in the running cost. In the case of stochastic control problems, such discontinuities may essentially be ignored (with respect to the convergence of the numerical schemes) if the optimally controlled process induces a measure on path space under which the total cost is continuous w.p.1 (see the remark after Theorem 10.1.1). When the underlying processes are deterministic the situation is quite different. In this case, properly dealing with the discontinuity becomes the main focus of interest. Although the dynamics are deterministic, the natural formulation of the cost along the discontinuity involves a "randomization" of the costs on either side, and so one might expect probabilistic methods to be effective. We will see that the Markov chain method continues to work well and, in fact, provides a very natural way to deal with a difficult problem for which there are no alternative methods.

Many of the complicating features we will discuss can also occur in standard stochastic and deterministic control problems. The interested reader can combine the methods used in this chapter and the next with those introduced previously to treat some of these generalizations.

An outline of the chapter is as follows. In Section 14.1 we consider a fixed time interval and suppose that the cost is the sum of a continuous running cost and a terminal cost. This is the calculus of variations analogue of the control problem treated in Chapter 12, and is known as a Bolza problem. In Section 14.2 we describe the numerical schemes and give the proof of convergence. Problems where the running cost is discontinuous in the state variable are considered Section 14.3. Problems with a controlled stopping time are the topic of Chapter 15.

14.1 Problems with a Continuous Running Cost

Let \( k : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R} \) denote a running cost. We will assume throughout this chapter and the next that \( k(\cdot, \cdot) \) satisfies the following uniform superlinear growth condition:

\[
\lim \inf_{c \to \infty} \inf_{x, \alpha : |\alpha| = c} k(x, \alpha)/c = +\infty. \tag{1.1}
\]

This condition is natural and holds in most applications. Under (1.1) there exists a convex function \( l : [0, \infty) \to (-\infty, +\infty] \) which is bounded from below and satisfies

\[
k(x, \alpha) \geq l(|\alpha|) \text{ for all } (x, \alpha), \text{ and } \lim_{c \to \infty} l(c)/c \to \infty. \tag{1.2}
\]