Continuous Optimal Control
Sensitivity Analysis with AD

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ABSTRACT In order to apply a parametric method to a minimum time control problem in celestial mechanics, a sensitivity analysis is performed. The analysis is continuous in the sense that it is done in the infinite dimensional control setting. The resulting sufficient second order condition is evaluated by means of automatic differentiation, while the associated sensitivity derivative is computed by continuous reverse differentiation. The numerical results are given for several examples of orbit transfer, also illustrating the advantages of automatic differentiation over finite differences for the computation of gradients on the discretized problem.

11.1 Introduction

This chapter is concerned with the use of automatic differentiation (AD) in the context of sensitivity analysis of optimal control problems (here, the minimum time transfer of a satellite to a geostationary orbit [107, 404]). Whereas AD is commonly employed on approximated optimal control problems [117], it is seldom used before discretization, in the infinite dimensional setting typical of control. The originality of this article is a use of AD not only to compute gradients of the discretized problem, but also to perform a continuous sensitivity analysis (see also [90] in the case of PDEs). AD then turns to be an efficient way to deal with the cumbersome computations involved in real-life control problems.

The minimum time orbit transfer problem is briefly stated in §11.2. Then, an outline of the specific parametric technique developed to solve it is presented in §11.3; its use requires the sensitivity analysis of interest here, which essentially amounts to integrating a Riccati equation evaluated by AD. The associated sensitivity derivative is computed by reverse differentiation. Some numerical results for the orbit transfer are given in §11.4, especially for very low thrust transfers. Besides, they demonstrate the relevance of AD to evaluate the gradients of the discrete algorithm.
11.2 Low Thrust Orbit Transfer

The problem motivating this study is the minimum time transfer of a satellite towards a geostationary orbit. The dynamics is expressed using the orbital parameters that define the ellipse osculating to the trajectory (since these coordinates are first integrals of the unperturbed motion, they are slowly varying. On the other hand, the expression of the dynamics becomes intricate). We use a more realistic model than in [107], taking into account the variation of the mass \( m \), so that, on a suitable open submanifold of \( \mathbb{R}^n \) (\( n \) is the dimension of the system; \( n = 4 \) for the 2D model, \( n = 6 \) for the 3D one), the dynamics can be written as:

\[
\begin{align*}
\dot{x} &= f_0(x) + B(x)u/m \\
\dot{m} &= -\delta|u|,
\end{align*}
\]

where the control \( u \) is the thrust of the engine, and \(|.|\) is the Euclidean norm (see [107, 404] for more details). There are also boundary conditions defining the initial and the final orbit,

\[
x(0) = x^0, \quad m(0) = m^0, \quad h(x(t_f)) = 0,
\]

together with a constraint on the maximum modulus of the thrust:

\[
|u| \leq F_{\text{max}}
\]

with \( F_{\text{max}} \) small (low thrust transfer). The problem of finding an absolutely continuous state \((x, m)\), and an essentially bounded control \( u \) that minimize the transfer time \( t_f \) will be referred to as \((SP)_{F_{\text{max}}}\). Among other results, it is proven in [104] that any optimal control has finitely many switchings so that \(|u| = F_{\text{max}}\) almost everywhere. As a consequence, \( m(t) = m^0 - \delta F_{\text{max}} t \) and \((SP)_{F_{\text{max}}} \) is reduced to a non-autonomous problem. The technique used to solve it is described in the next section.

11.3 Continuous Sensitivity Analysis

Rather than using direct methods (e.g., direct transcription) that lead to nonlinear programming, we emphasize indirect approaches. They are faster and more accurate for our problem (see [103, 105, 347] for comparisons). Their main drawback is the loss of robustness: the sensitivity of single shooting to the initialization of the adjoint state is well-known [16]. In an attempt to deal with these difficulties, a new parametric technique is introduced in [107] for minimum time control problems. We here give an outline of the method for the problem of §11.2. If we denote by \( \phi(\beta) \) the value function of the optimal control problem \((SP)^{\beta}_{F_{\text{max}}} \) with fixed final time \( \beta \) (reformulated for convenience on \([0,1]\) with an obvious change of