Nonlinear Observer Design Using Automatic Differentiation

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ABSTRACT State feedback controllers are widely used in real-world applications. These controllers require the values of the state of the plant to be controlled. An observer can provide these values. For nonlinear systems there are some observer design methods which are based on differential geometric or differential algebraic concepts. The application to non-trivial systems is limited due to a burden of symbolic computations involved. The authors propose an observer design method using automatic differentiation.

15.1 Introduction

During the last few years, researchers have developed many algorithms for controller and observer design for nonlinear state-space systems

\[ \dot{x}(t) = f(x(t)) + g(x(t)) \cdot u(t), \quad y(t) = h(x(t)). \] (15.1)

The vector-valued state is denoted by \( x \), \( u \) denotes the scalar input signal, and \( y \) denotes the scalar output signal. Some design methods are based on differential geometric or differential algebraic concepts [195, 302]. These approaches are based on time-consuming symbolic computations. We show how the use of automatic differentiation (AD) avoids cumbersome computations.

State feedback control requires the value of the state \( x \). An observer is a dynamical system, whose state \( \hat{x} \) asymptotically tracks the state \( x \) of the original system (15.1). As for general nonlinear systems, observer design is still an active area of research [447]. To simplify presentation, this chapter considers the unforced system case. In this case, several observer design methods are known [65]. Their application to non-trivial systems is limited due to the symbolic computations involved [154, 443]. The authors propose an observer design method based on automatic differentiation which is applicable to single output systems.

In §15.2, we review observer design for linear time-varying systems. The use of AD for the design of nonlinear observers is described in §15.3. The method derived there will be applied in §15.4 to an example.
15.2 Theory

In this section, we summarize some results of observer design. Consider a linear time-varying system

\[ \dot{x}(t) = A(t) x(t), \quad y(t) = C(t) x(t) \quad (15.2) \]

with a scalar-valued output \( y \). For an observer

\[ \dot{x}(t) = A(t) \tilde{x}(t) + k(t) (y(t) - C(t) \tilde{x}(t)) , \]

we want to compute the gain vector \( k(t) \) in such a way that the eigenvalues of the observer are assigned to desired places. This task is straightforward if the system matrices were given in observer canonical form...

\[ \tilde{A}(t) = \begin{pmatrix} 0 & \cdots & 0 & -a_0(t) \\ 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 1 & -a_{n-1}(t) \end{pmatrix}, \quad \tilde{C}(t) = (0 \cdots 0 1). \quad (15.3) \]

Then, the eigenvalues to be assigned are the roots of the characteristic polynomial

\[ \det(sI - \tilde{A}(t) + \tilde{k}(t)\tilde{C}(t)) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0, \]

where the coefficients \( p_0, \ldots, p_{n-1} \) are constant, and the components of the vector \( \tilde{k}(t) = (\tilde{k}_1(t), \ldots, \tilde{k}_n(t))^T \) are \( \tilde{k}_i(t) = p_{i-1} - a_{i-1}(t) \) for \( i = 1, \ldots, n \).

To get \( k(t) \) for the general case, a well-known generalization of Ackermann's formula for linear time-varying systems is used here \[65, 196, 447]. The observer canonical form (15.3) can be obtained from (15.2) by a change of coordinates \( \tilde{x}(t) = T(t) x(t) \) if the observability matrix

\[ Q(t) := \begin{pmatrix} C(t) \\ \mathcal{L}C(t) \\ \vdots \\ \mathcal{L}^{n-1}C(t) \end{pmatrix} \quad (15.4) \]

has full rank. The differential operator \( \mathcal{L} \) in (15.4) is defined by

\[ \mathcal{L}C(t) = \dot{\tilde{C}}(t) + C(t) A(t). \quad (15.5) \]

The inverse transformation matrix \( T^{-1} \) can be computed as \( T^{-1}(t) = (q(t), \mathcal{L}q(t), \ldots, \mathcal{L}^{n-1}q(t)) \), where \( q \) denotes the last column of the inverse observability matrix, i.e., \( q(t) = Q^{-1}(t) (0, \ldots, 0, 1)^T \), and the differential operator \( \mathcal{L} \) is defined by

\[ \mathcal{L}q(t) = -q(t) + A(t) q(t). \quad (15.6) \]

Finally, the desired gain vector \( k(t) \) can be calculated as follows:

\[ k(t) = \left( p_0 I_n + p_1 \mathcal{L} + \cdots + p_{n-1} \mathcal{L}^{n-1} + \mathcal{L}^n \right) q(t). \quad (15.7) \]