15 Fibrations, homotopy groups and Lie group actions

In this section the ground ring is an arbitrary field \( k \) of characteristic zero.

In this section we see how relative Sullivan algebras model fibrations. In particular, if \( f : X \to Y \) is a continuous map with homotopy fibre \( F \) we construct a Sullivan model for \( F \) directly from the morphism \( A_{PL}(f) : A_{PL}(Y) \to A_{PL}(X) \), provided \( Y \) is simply connected with rational homology of finite type.

We use this to construct Sullivan models for many more interesting spaces. We are also able, at last, to establish the isomorphism

\[
V \cong \text{Hom}_\mathbb{Z}(\pi_*(X), k)
\]

(promised in §12 and §13(c)) between the generators of a minimal model and the dual of the homotopy groups.

Finally we apply Sullivan models to the study of principal bundles and group actions of a path connected topological group \( G \). Our main focus is on the case when \( H_*(G; k) \) is finite dimensional, which includes all connected Lie groups. In particular, we use Milnor's universal bundle (§2) to obtain a simple form for the Sullivan model of any principal bundle. We also consider models for group actions and see, for example, rational homotopy reasons why spaces may not support free actions of groups such as \( S^3 \times SU(3) \).

Much of the material was originally developed by H. Cartan, Koszul and Weil in the context of smooth principal bundles, with the aid of principal connections and the curvature tensor. This was described in three lectures [33] [34] [102] given in Brussels in 1949 and provided one of the major clues that led to Sullivan's introduction of minimal models. Indeed, the main result announced in Koszul's lecture is the construction of (what we now call) a Sullivan model for a homogeneous space.

This section is organized into the following topics:

(a) Models of fibrations.

(b) Loops on spheres, Eilenberg-MacLane spaces and spherical fibrations.

(c) Pullbacks and maps of fibrations.

(d) Homotopy groups.

(e) The long exact homotopy sequence.

(f) Principal bundles, homogeneous spaces and Lie group actions.

(a) Models of fibrations.

Consider a Serre fibration of path connected spaces

\[
p : X \to Y,
\]
whose fibres are also path connected. Let \( j : F \to X \) be the inclusion of the fibre at \( y_0 \in Y \). Then \( A_{PL} \) converts the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{j} & X \\
\downarrow & & \downarrow p \\
y_0 & \xrightarrow{} & Y
\end{array}
\]

\[
A_{PL}(F) \xleftarrow{A_{PL}(j)} A_{PL}(X)
\]

where \( \epsilon \) is the augmentation corresponding to \( y_0 \).

Observe that \( H^1(A_{PL}(p)) \) is injective. Indeed, since \( F \) is path connected it follows from the long exact homotopy sequence that \( \pi_1(p) \) is surjective (Proposition 2.2). Hence \( H_1(p; \mathbb{Z}) \) is surjective (Theorem 4.19). But

\[
H_1(Y; \mathbb{K}) = H_1(Y; \mathbb{Q}) \otimes \mathbb{K} = H_1(Y; \mathbb{Z}) \otimes \mathbb{K},
\]

and so \( H_1(p; \mathbb{K}) \) is also surjective. Thus the dual map, \( H^1(A_{PL}(p)) = H^1(p; \mathbb{K}) \) is injective (Proposition 5.3(i)).

Since \( H^1(A_{PL}(p)) \) is injective, Proposition 14.3 asserts the existence of a Sullivan model for \( p \),

\[
m : (A_{PL}(Y) \otimes \Lambda V, d) \overset{\sim}{\to} A_{PL}(X).
\]

(In fact (Theorem 14.12) there is even a minimal Sullivan model for \( p \).)

The augmentation \( \epsilon : A_{PL}(Y) \to \mathbb{K} \) defines a quotient Sullivan algebra

\[
(\Lambda V, \overline{d}) = \mathbb{K} \otimes_{A_{PL}(Y)} (A_{PL}(Y) \otimes \Lambda V, d),
\]

which is called the fibre of the model at \( y_0 \).

Since \( A_{PL}(j)A_{PL}(p) \) reduces to \( \epsilon \) in \( A_{PL}(Y) \), \( A_{PL}(j)m \) factors over \( \epsilon \cdot id \) to give the commutative diagram of cochain algebra morphisms

\[
\begin{array}{ccc}
A_{PL}(F) & \xleftarrow{A_{PL}(j)} & A_{PL}(X) \\
\overline{m} & \uparrow & \uparrow m \\
(\Lambda V, \overline{d}) & \xleftarrow{\epsilon \cdot id} & (A_{PL}(Y) \otimes \Lambda V, d)
\end{array}
\]

We show now that, under mild hypotheses, \( \overline{m} \) is a quasi-isomorphism. Thus in this case \( \overline{m} : (\Lambda V, \overline{d}) \overset{\sim}{\to} A_{PL}(F) \) is a Sullivan model for \( F \): the fibre of a model is a model of the fibre.

**Theorem 15.3** Suppose \( Y \) is simply connected and one of the graded spaces \( H_*(Y; \mathbb{K}) \), \( H_*(F; \mathbb{K}) \) has finite type. Then

\[
\overline{m} : (\Lambda V, \overline{d}) \to A_{PL}(F)
\]

is a quasi-isomorphism.