7

Weighted Polynomial Approximation

7.1 Statement of Results

In this chapter, we establish the existence of weighted polynomial approximations that are a prerequisite to the estimates and asymptotics in subsequent chapters. We search for polynomials \( P_n \) of degree \( n \) such that \( P_n W \) approximates 1 in some sense on \([a_{-n}, a_n]\). There are several approaches to this problem ([48], [86], [115], [136], [166], [178]), most based on discretisation of the potential. The most successful one has been given by Totik [178], and this is the method that we employ. Because of our need to approximate almost up to \( a_{\pm n} \), our results do not follow from those in [178], [180], [182].

We shall present two types of results: Theorems 7.1 and 7.2, where we approximate 1 by \( R_n W \) in a uniform sense; and Theorems 7.3 and 7.4, where we require only \( R_n W \sim 1 \) in a suitable interval. We shall also need another type of approximation, namely in a geometric mean sense. We delay this to the next chapter.

**Theorem 7.1**

Let \( W \in \mathcal{F} \) (dini) and let \( 0 < \alpha < \beta \).

(a) There exists \( C > 0 \) and for \( n \geq 1 \) polynomials \( R_n \) of degree \( \leq \beta n \), positive on the real line, such that

\[
\| R_n W - 1 \|_{L_\infty(\Delta_n)} = o(1), \quad n \to \infty
\]

and

\[
\| R_n W \|_{L_\infty(I)} \leq 1.
\]
(b) If also \( W \in \mathcal{F}(\psi) \) with \( \psi(s) = s^\delta \), some \( \delta > 0 \), then one may replace \( \Delta_{an} \) by \( [a_{-\beta n}(1 - Cn^{-\eta}), a_{\beta n}(1 - Cn^{-\eta})] \) and \( o(1) \) by \( Cn^{-\eta} \) for some \( \eta > 0 \) small enough.

**Remark**
The restriction \( \alpha < \beta \) is essential. It can be shown (cf. [76], [117]) that, in general, no sequence \( \{R_n\} \) with \( \deg(R_n) \leq \beta n \), can satisfy (7.1) with \( \Delta_{an} \) replaced by \( \Delta_{\beta n} \). However, one can come close, as part (b) shows.

It will be convenient to restate Theorem 7.1 in “contracted” form. Let \( L_n \) denote the linear map of \( \Delta_n \) onto \([-1, 1]\) and let \( L_n^{-1} \) denote its inverse. Define

\[
W_n^*(t) := W\left(L_n^{-1}(t)\right). 
\]

Next, given \( \varepsilon > 0 \), consider the interval

\[
\Delta_n^*(\varepsilon) := [-1 + \varepsilon\chi_{-n}, 1 - \varepsilon\chi_{n}],
\]

where

\[
\chi_{\pm n} := \frac{|a_{\pm n}|}{\delta_n T(a_{\pm n})} < 2
\]

(recall that \( T \geq \Lambda > 1 \)). Note that by Lemma 3.6(a),

\[
L_n(a_{\alpha n}) = \frac{a_{\alpha n} - \beta_n}{\delta_n} = 1 - \frac{a_n - a_{\alpha n}}{\delta_n} 
\leq 1 - C\frac{a_n}{\delta_n T(a_n)} (1 - \alpha)^2 = 1 - C\chi_n (1 - \alpha)^2,
\]

uniformly for \( n \geq 1 \), \( \alpha \in (\frac{1}{2}, 1) \), with a similar relation for \( a_{-\alpha n} \). Therefore, for some \( C \neq C(\alpha, n) \),

\[
\Delta_n^* \left( C(1 - \alpha)^2 \right) \supset L_n(\Delta_{\alpha n}).
\]

With this in mind we shall easily deduce Theorem 7.1 from:

**Theorem 7.2**

Let \( W \in \mathcal{F}(\text{dini}) \) and let \( W_n^* \) be as above.

(a) There exists a sequence \( \{d_n\} \) increasing to \( \infty \), a constant \( C \), and for \( n \geq 1 \) polynomials \( R_n^* \) of degree \( \leq n \), positive on the real line, such that

\[
\|R_n^*W_n^* - 1\|_{L_\infty(I_n)} \leq Cd_n^{-1}, \quad n \to \infty,
\]

where

\[
I_n := [-1 + \chi_{-n}d_n^{-1}, 1 - \chi_n d_n^{-1}] = \Delta_n^* (d_n^{-1}).
\]