A GENERALIZED EIGENFUNCTION EXPANSION FOR ELASTODYNAMICS

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INTRODUCTION

The use of composite materials in structural components is limited, in part, by the need to develop quantitative inspection techniques. Composites present many challenges to the development of such quantitative methods. For example, composite materials are typically anisotropic. Thus a necessary prerequisite for ultrasonic flaw detection and characterization in composite materials is an understanding of the propagation and scattering of waves in a general anisotropic media. Work towards such an understanding has been typically limited to such issues as beam propagation effects (see, for ex., Ref. [1]). The elastic wave inverse scattering problem (flaw characterization), as well as the simpler direct problem (field-flaw interaction), are only minimally developed for anisotropic media of any kind.

In this paper we review a study of the elastic wave inverse scattering problem in a general anisotropic media (Refs. [2], [3], and [4]). Two basic results are presented, which are expected to be of considerable use. First, we present a representation for the spatial Dirac delta function, $\delta(x - y)$, in terms of a product of two near-field point-source scattering solutions. Second, the far-field asymptotics of this delta-function representation yields the generator for a generalized eigenfunction expansion for anisotropic elastodynamics. This expansion allows the reconstruction of a function from its projections onto wavefields which evolve from plane-wave incidence on an inhomogeneous scatterer.

It is important to note that both the delta-function representation and the generalized eigenfunction expansion contain a function $\tau(x, y)$, which is at the disposal of the user. We note that this greatly expands the traditional form and uses of eigenfunction expansions. The role of these expansions in inverse scattering theory will be discussed in the sequel.
EIGENFUNCTION EXPANSIONS

The concept and utility of eigenfunction expansions is best illustrated by an example. The eigenfunctions of the Laplacian operator over $\mathbb{R}^3$ are the exponentials satisfying

$$(\nabla^2 + k^2)e^{\pm ik \cdot x} = 0 .$$

(1a)

Here the eigenvalue $k$ is defined as the positive root of

$$k = (k_i k_i)^{1/2} ,$$

(1b)

where $k_i$ is the $i^{th}$ component of the vector $k$. Roman indices take on the integer values 1-3, and the summation convention over repeated indices is assumed throughout. The statement of orthogonality of the eigenfunctions is

$$\delta(x - y) = (2\pi)^{-3} \int d^3k e^{ik \cdot x} e^{-ik \cdot y} ,$$

(2)

where the integral operator $\int d^3k$ indicates integration over all of $k$-space.

Equation (2) is a generator for the Fourier transform of a function $f(x)$. To demonstrate, we operate on both sides of Eq.(2) with $\int d^3y f(y)$. This gives:

$$f(x) = (2\pi)^{-3} \int d^3k e^{ik \cdot x} \int d^3y e^{-ik \cdot y} f(y) .$$

(3)

Let us denote the function generated by the $y$-integration in Eq. (3) as $\hat{f}(k)$. This function, commonly called the Fourier transform of $f(x)$, can be thought of as the projection of $f(y)$ on the eigenfunctions of the Laplacian operator, see Eq. (1). Because of the delta-function on the left-hand side of Eq. (2), the $k$-integration in Eq. (3) ensures the reconstruction of $\hat{f}(k)$ to the original $f(x)$.

Note that the eigenfunctions of Eq. (1) have the physical interpretation of time-harmonic ($e^{-i\omega t}$) plane waves, freely propagating in an infinite acoustic medium with unit velocity. Thus expression (3) allows reconstruction of a function from its projection onto these plane waves. Of interest to inverse scattering theory are equivalent reconstruction formalisms where other physically meaningful functions play the role of the plane waves as basis functions. It has been shown (Refs [5], [6]) that the solutions to the acoustic wave equation in the presence of an inhomogeneity provide such a set of basis functions. In particular, it is possible to define a generalized Fourier transform in terms of this expansion. An important consequence (Refs. [5], [6]) is that the velocity and scattering amplitude are transform pairs, in the sense of such a generalized Fourier transform.

In this paper we demonstrate that another set of basis functions, which provide an even further generalization of the Fourier transform, are those solutions to the general anisotropic elastic wave equation in the presence of an inhomogeneity.