We saw in the previous chapter several representations embodied in the algebra (4.10)-(4.12). The most economical one is obtained in the bosonic string by writing

\[ x^\mu(\tau, \sigma) = x^\mu(\tau - \sigma) + \tilde{x}^\mu(\tau + \sigma), \]

and similarly for \( p^\mu \).

The algebra then splits into one piece from the right-moving coordinates \( x^\mu(\tau - \sigma), p^\mu(\tau - \sigma) \) and one from the left-moving part. The two parts do not mix and we can freely set one of them to zero. Similarly for the full algebra (4.10)-(4.12) it is straightforward to see that for the terms where \( S^A \) couple to \( x \) and \( p \), \( S^1 \) couple to the right-moving and \( S^2 \) to the left-moving parts. Since they are right-moving and left-moving, respectively, again a consistent truncation can be made by setting, say, the left-moving parts equal to zero. This reduces the algebra to an \( N = 1 \) supersymmetry.

The heterotic string [30] is now constructed by putting together the algebra from one right-moving superstring constructed as above with the algebra from a 26-dimensional left-moving bosonic string. There is an obvious mismatch in dimensions! The solution to this dilemma is that we only add in the \( \text{SO}(1,9) \) subalgebra of the full bosonic algebra. Eventually we must interpret the extra coordinates \( x^I(\tau, \sigma), I = 1, \ldots, 16 \) and check what has happened to the \( \text{SO}(16) \) symmetry left out.

We now check the algebra closer. Consider the bosonic algebra. For a left-moving string

\[ p^-= \frac{1}{4\pi p^+} \int_0^\pi d\sigma (\pi p^I + \dot{x}^I)^2 \]

\[ (5.1) \]
The combination $\pi p^I + \dot{x}^I$ is only left-moving. If we now insist that $p^I$ and $x^I$ both separately are left-moving, we see in the mode expansion that they are the same (up to a $\pi$) and we have to change the canonical commutator to

$$[x^I(\sigma, \tau), p^J(\sigma', \tau)] = \frac{i}{2} \delta(\sigma - \sigma') \delta^{IJ}$$  \hspace{1cm} (5.2)$$

and impose

$$[x^I(\sigma, \tau), x^J(\sigma, \tau)] = -\frac{i}{2} \pi \frac{1}{\partial \sigma} \delta(\sigma - \sigma') \delta^{IJ}$$ \hspace{1cm} (5.3)$$

where

$$\frac{1}{\partial \sigma} \delta(\sigma - \sigma') = \epsilon(\sigma - \sigma')$$

This change does not alter the closure of the rest of the algebra. Hence in the full algebra of the heterotic string

$$p^- = \frac{1}{2\pi p^+} \int_0^\pi d\sigma [\pi^2 p^i + \dot{x}^i - i S^a \dot{S}^a + 2\pi p^I \dot{x}^I]$$ \hspace{1cm} (5.4)$$

and the constraint on $x^-(\sigma)$ is

$$\dot{x}^-(\sigma) = \frac{\pi}{p^-} p^i \dot{x}^i + \frac{i}{2p^+} S^a \dot{S}^a + \frac{\pi}{p^-} p^I \dot{x}^I$$ \hspace{1cm} (5.5)$$

with $S = S^1$. The remaining generators of the $N = 1$ super-Poincaré algebra are obtained by putting $S^2 = 0$ in algebras (4.10)-(4.12).

The action corresponding to the Hamiltonian (5.4) is

$$S = -\frac{T}{2} \int_0^\pi d\sigma \int d\tau \left[ \eta_{\alpha\beta}(\partial_\alpha x^I \partial_\beta x^I + \partial_\alpha x^I \partial_\beta x^I) + i S^a \left( \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} \right) S^a \right]$$ \hspace{1cm} (5.6)$$

together with constraints

$$\Phi^I = \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} \right) x^I = 0$$ \hspace{1cm} (5.7)$$