FOUR-LOOP SIGMA-MODEL BETA-FUNCTIONS VERSUS $\alpha'^{3}$ CORRECTIONS TO SUPERSTRING

EFFECTIVE ACTIONS

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ABSTRACT

We present results on higher-order calculations for the N=1 supersymmetric $\sigma$-model in two dimensions. The condition of vanishing four-loop $\beta$-function, interpreted as the classical equations of motion of the associated string theory, can be derived from an effective action. It represents the generalization to an arbitrary Riemannian manifold of the corresponding low-energy action obtained from the four-graviton scattering amplitude for type II superstring. Thus the $\sigma$-model approach indeed describes the tree-level dynamics of the superstring.

The low-energy physics of string theories can be described by a local field approximation as an expansion in the inverse string tension $\alpha'$. In this limit one obtains an effective field theory for the massless fields of the string with all the massive modes integrated out. The corresponding action, which is given by an infinite sum of terms with increasing number of derivatives, generates all the string tree-level amplitudes of the massless particles. The motivations for computing this effective action reside primarily in the attempt to derive consistent string compactifications and viable phenomenology. Moreover one can hope that a knowledge of higher derivative terms in the expansion will give insights on general invariance principles underlying the full string theory. While a priori straightforward, the actual procedure for such a construction requires considerable effort.

It has recently become clear that an alternative way to get information about the string low-energy effective theory is provided by the $\sigma$-model approach. The fact that non-linear $\sigma$-models in two dimensions are strictly related to tree string S-matrix elements can be easily understood. Let us consider for simplicity the purely gravitational bosonic $\sigma$-model. The action is

$$S = (1/4\pi\alpha') \int d^2 x g_{ij}(\phi) \partial^\mu \phi^i \partial^\nu \phi^j$$

(1)

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where $\phi^i$ are the spacetime coordinates and $g_{ij}$ is the metric tensor. If one splits the metric around a flat background $g_{ij} = \delta_{ij} + h_{ij}$ and expands the exponential in the functional integral, one obtains

$$Z = \int D\Phi \exp S$$

$$= \sum \frac{1}{n!} \int D\phi \left[ \frac{1}{4\pi \alpha'} \int d^2 x \ h_{ij}(\phi) \phi^{i} \phi^{j} \right] \exp S_0$$

where

$$S_0 = \left( \frac{1}{4\pi \alpha'} \right) \int d^2 x \ \delta^{\mu \nu} \ \delta_{ij} \ e^{ik_{1} \phi} \ e^{ik_{n} \phi} > S_0$$

is the free string action. By taking a momentum Fourier-transform and imposing on-shell conditions, one is led to consider the following expression

$$\sum \frac{1}{n!} \prod_{r} \left( \frac{\tilde{h}_{ij}}{\alpha'} \right) <\phi^{i_1} \phi^{j_1} \phi^{i_n} \phi^{j_n} > S_0$$

which reproduces the $n$-point scattering amplitudes for the graviton field of the string. More generally, one expands $g_{ij} = g_{ij}^0 + h_{ij}$ where $g_{ij}^0$ is a classical configuration and $h_{ij}$ is the on-shell field. Thus the $\sigma$-model can be viewed as the generating functional of string S-matrices\(^5,6\).

Indeed, it has been shown that loop graphs of the two-dimensional theory rearrange themselves to give tree graphs of the string\(^7\). In particular, the $\sigma$-model $\beta$-function becomes the tadpole amplitude of the string. As in ordinary field theory, the tree-level tadpole amplitude corresponds to the classical Euler-Lagrange equations. Therefore a knowledge of the $\beta$-function of the $\sigma$-model implies a knowledge of the equations of motion of the massless fields of the string. (An equivalent, complementary way to obtain the condition of vanishing $\beta$-function is to demand conformal invariance of the two-dimensional theory\(^3,4\).)

It is well-known that the one-loop gravitational $\beta$-function for the non-linear $\sigma$-model gives the correct equations of motion of the Einstein-Hilbert action. It corresponds to the zero-order term in the $\alpha'$ expansion of the effective action and as such it does not contain "string" physics. In order to include string effects one has to go beyond the leading approximation.

I will report on recent calculations of higher-order corrections to the $\beta$-function for supersymmetric non-linear $\sigma$-models\(^8\). Supersymmetry enters here since our aim is to make contact with superstring theories. More precisely, I will consider the purely metric $\sigma$-model with $N=1$ supersymmetry (the antisymmetric tensor field is set equal to zero and the dilaton is treated as a constant field). For this model, the two- and three-loop corrections to the $\beta$-function are identically zero\(^9,10,8\). The next-leading contribution arises at four loops\(^8\). A superfield formulation and supergraph techniques are essential tools to perform such a high-order perturbative calculation. The four-loop correction to the $N=1$ gravitational $\beta$-function is given by a rather complicated expression involving structures of conformal weight three in the Riemann tensors.

According to the superstring-$\sigma$-model correspondence, the equations $\beta=0$ should be derivable from an action that, as explained above, should be identified with the low-energy effective theory for the graviton field of the string. Indeed the action whose variation reproduces the equations $\beta=0$