Abstract—This chapter considers the number of gates to realize logic functions by OR-AND-OR three-level networks under the condition that both true and complemented variables are available, and each gate has no fan-in and fan-out constraints. We show that an arbitrary \( n \)-variable function can be realized by an OR-AND-OR three-level network with at most \( 2^{r+1} + 1 \) gates, where \( n = 2^r \) and \( r \) is an integer. We also prove that for sufficiently large \( n \), regardless of the number of levels, we need at least \( 2^{r+1}(1-\xi) \) gates to realize almost all functions of \( n \) variables by an AND-OR multi-level network, where \( \xi \) is an arbitrarily small positive real number \( (0 < \xi < 1) \). We developed a heuristic algorithm to design OR-AND-OR three-level networks, realized various functions, and compared the number of gates for OR-AND-OR three-level networks with AND-OR two-level ones. For arithmetic functions of 8 variables, three-level networks require, on the average, 40\% fewer gates than AND-OR two-level ones. For other benchmark functions of 9 to 128 variables, three-level networks required up to 91\% fewer gates. For randomly generated functions of 10 variables, three-level networks required 50\% fewer gates.

13.1 INTRODUCTION

Representing logic functions by using logic gates is a fundamental problem in logic design. Optimization methods for AND-OR two-level networks are well established [6].
Lawler [5] considered a design method for multi-level networks, such as OR-AND-OR, AND-OR-AND-OR, \ldots, etc. He showed a systematic way to find simple networks.

OR-AND-OR three-level networks are interesting research topics, since they require many fewer gates than AND-OR two-level ones, and are relatively easy to design.

Bounded-depth networks [1, 3] are desirable since the propagation delays of them are easy to estimate. Also, they can be used as initial networks for multi-level logic networks with fan-in limited gates.

In this chapter, we consider the number of gates to realize arbitrary functions by AND-OR multi-level networks under the condition that both true and complemented variables are available as inputs, and each gate has unlimited fan-in and fan-out.

An arbitrary logic function can be realized by an AND-OR two-level network. In this case, $2^{n-1} + 1$ gates are necessary and sufficient to realize a parity function of $n$ variables, which requires the largest number of gates in two-level realizations [13, 16]. In a similar way, an arbitrary function can be realized by multi-level networks such as OR-AND-OR three-level networks, AND-OR-AND-OR four-level networks, etc. Then, how many gates are necessary to realize an arbitrary function by a multi-level network? And, which function requires the largest number of gates in a multi-level realization? This chapter considers the number of gates to realize arbitrary functions and tries to find the most complex function in multi-level networks.

First, we show that an arbitrary function of $n$ variables ($n = 2r$, where $r$ is an integer) can be realized by an OR-AND-OR three-level network by using at most $2^{r+1} + 1$ gates. Second, we prove that, regardless of the number of levels, we need at least $2^{r+1}(1 - \xi)$ gates to realize almost all functions of $n$ variables by multi-level AND-OR networks, where $n$ is a sufficiently large integer, and $\xi$ is an arbitrarily small real positive number ($0 < \xi < 1$). Thus, an AND-OR two-level network requires $2^{n-1} + 1$ gates in the worst case, while an OR-AND-OR three-level network requires at most $2^{r+1} + 1$ gates, where $r = n/2$. From these facts, we know that there is a distinct difference between AND-OR two-level networks and OR-AND-OR three-level ones, but not so much difference between three-level ones and multi-level ones with more than three levels.