Chapter 1

Optimization

Optimization is the art of finding the best among several alternatives in decision making.

Let S be the set of possible decisions. This set is called the *feasible set*. The *decision variable* is denoted by x. If $x \in S$, then x is called *feasible*, otherwise *infeasible*. The net costs caused by decision x are measured by a real valued *objective function* $F(x)$. The goal is to find the best decision, i.e. the decision with minimal costs. We will always assume that there is one single objective function. The case of several competing objective functions, the *multi-criteria decision* making problem will not be touched here.

The general form of an optimization problem (P) is

$$(P) \qquad \left\| \begin{array}{l} \text{Minimize } F(x) \\ x \in S \end{array} \right. \tag{1.1}$$

Without loss of generality we have stated here a *minimization problem* since trivially any *maximization problem* can be brought into the form (1.1) by considering $-F(x)$ instead of $F(x)$. What concerns the solution, one is interested in the *optimal value*

$$\inf\{F(x) : x \in S\}, \tag{1.2}$$

which exists if S is nonempty, and/or in the *set of optimal solutions*

$$\operatorname{argmin}\{F(x) : x \in S\} := \{x \in S : F(x) = \inf\{F(u) : u \in S\}\}, \tag{1.3}$$

which is nonempty if the infimum in (1.2) is not $-\infty$ and is attained. Every point $x^* \in \operatorname{argmin}\{F(x) : x \in S\}$ is called a *minimizer* of F. A point $x^+ \in S$

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is called *local minimizer* of F, if there is a neighborhood U of x^+, such that $x^+ \in \operatorname{argmin}\{F(x) : x \in S \cap U\}$.

We will mostly treat the case of a real or integer valued d-dimensional *decision space*, i.e.

$$S \subseteq \mathbb{R}^d \text{ or } S \subseteq \mathbb{Z}^d.$$

We speak of an *integer problem*, if $S \subseteq \mathbb{Z}^d$ and of an *unconstrained problem*, if $S = \mathbb{R}^d$. Occasionally we will also consider the case of a finite decision space $S = \{x_1, \ldots, x_m\}$.

Any constrained problem may be transformed into an unconstrained non-smooth problem by lifting the constraints to the objective function: Let $\vartheta_S(\cdot)$ be the function

$$\vartheta_S(x) = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{if } x \notin S \end{cases} \tag{1.4}$$

Then trivially the two programs

$$\left\| \begin{array}{l} \text{Minimize } F(x) \\ x \in S \end{array} \right. \qquad \text{and} \qquad \left\| \begin{array}{l} \text{Minimize } F(x) + \vartheta_S(x) \\ x \text{ arbitrary} \end{array} \right.$$

are just two different representations of the same optimization problem. Since ϑ_S is very nonsmooth, the second representation does not lead to new solution algorithms and is only of theoretical character. However, approximations to the second program are known as *penalty methods* and widely used (see section 1.4.8).

Constraints are typically given by a set of inequalities:

$$\begin{aligned} F_1(x) &\leq 0 \\ F_2(x) &\leq 0 \\ &\cdots \\ F_r(x) &\leq 0. \end{aligned} \tag{1.5}$$

Equality constraints $F_i(x) = 0$ can be split into two inequality constraints, namely $F_i(x) \leq 0$ and $-F_i(x) \leq 0$ to fit into this scheme. However, most of the solution algorithms treat equality and inequality constraints quite differently.

In a *deterministic optimization problem* the functions F and F_i are explicitly known. They may be linear, quadratic or convex to yield a *linear, quadratic* or *convex mathematical program.*