Chapter 5

Stochastic Approximation

This chapter deals with algorithms for the optimization of simulated systems. In particular we study stochastic variants of the gradient algorithm

$$x_{n+1} = x_n - a_n \nabla F(x_n) \tag{5.1}$$

which was introduced in (1.27) to solve the optimization problem

$$\left\| \begin{array}{l} \text{Minimize } F(x) \\ x \in \mathbb{R}^d \end{array} \right.$$

where F is bounded from below.

The stochastic version of (5.1) is needed in the case where the objective function $F(x)$ or its gradient $f(x) := \nabla F(x)$ can be observed only by computer simulation. Suppose that for each $x \in \mathbb{R}^d$ one may get an estimate $Y(x)$ of $f(x)$ which contains a deterministic error $R(x)$ and a zero-mean random error $W(x)$

$$Y(x) = f(x) + R(x) + W(x).$$

The systematic error $R(x)$ contains the bias of $Y(x)$ in situations where an unbiased observation of $f(x)$ is impossible. Observations are biased in particular in the following two cases:

- **The Kiefer–Wolfowitz (KW-)procedure:** This procedure applies in situations when there are unbiased estimates $Z(x)$ of $F(x)$ available, but not such estimates of $f(x)$. Define the *finite-difference* estimate (here for

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simplicity in the one–dimensional case, see (5.9) for the multidimensional situation) as

$$\begin{aligned} Y(x) &= \frac{Z(x+c) - Z(x-c)}{2c} \\ &= f(x) + R(x) + W(x) \qquad \text{(say).} \end{aligned}$$

The systematic error of $Y(x)$ is $R(x) = \frac{F(x+c)-F(x-c)}{2c} - f(x)$ which is small if c is small. The zero–mean random error

$$W(x) = \frac{Z(x+c) - F(x+c) - Z(x-c) + F(x-c)}{2c}$$

has unbounded variance if c tends to zero and $Z(x+c)$ is independent from $Z(x-c)$. The right choice for c as a sequence of constants tending to zero (but not too fast !) is crucial for the KW-procedure (see subsection 5.1.1). Coupling $Z(x+c)$ and $Z(x-c)$ is another method of contolling the variance.

- **Optimization of Markov or Semi–Markov processes:** If $f(x)$ is the gradient of a performance function which is the steady–state expectation of a Markov process, then the finite horizon estimate of f contains a systematic error, which – by ergodicity – decreases to zero as the simulation length goes to infinity (see section 4.3.3).

The stochastic generalization of the gradient method is based on the recursion

$$X_{n+1} = X_n - a_n(f(X_n) + R_n + W_n), \tag{5.2}$$

where $R_n = R(X_n)$ and $W_n = W(X_n)$.

5.1 Convergence and asymptotic distributions

The first limit result will be stated in a pure deterministic setting. After that we discuss probabilistic models which lead to an almost sure convergence result. The method of proving the first theorem is called *Ljapunov's first method* whereas Theorem 3 uses *Ljapunov's second method* and martingales.

5.1 Theorem. Let (X_n) be a recursion of the form (5.2). Suppose that

(i) f fulfills a Lipschitz condition $||f(x) - f(y)|| \le K||x - y||$,

(ii) $\sum a_n ||R_n||^2 < \infty$,