12 POSTOPTIMALITY ANALYSIS II

In the preceding chapter we considered, in part, an assortment of postoptimality problems involving discrete changes in only selected components of the matrices $C$, $b$, or $A$. Therein emphasis was placed upon the extent to which a given problem may be modified without breaching its feasibility or optimality. We now wish to extend this sensitivity analysis a bit further to what is called parametric analysis. That is, instead of just determining the amount by which a few individual components of the aforementioned matrices may be altered in some particular way before the feasibility or optimality of the current solution is violated, let us generate a sequence of basic solutions which in turn become optimal, one after the other, as all of the components of $C$, $b$, or a column of $A$ vary continuously in some prescribed direction. In this regard, the following parametric analysis will involve a marriage between sensitivity analysis and simplex pivoting.

12.1. Parametric Analysis

If we seek to

$$\max f(X) = C'X \quad \text{s.t.} \quad AX = b, \; X \geq 0,$$

then an optimal basic feasible solution emerges if $X_B = B^{-1}b \geq 0$ and $C'_R - C'_BB^{-1}R \leq 0'$ (or, in terms of the optimal simplex matrix, $C'_BB^{-1}R - C'_R \geq 0'$) with $f(X) = C'_BX_B = C'_BB^{-1}b$. Given this result as our starting point, let us examine the process of...
(1) PARAMETRIZING THE OBJECTIVE FUNCTION. Given the above optimal basic feasible solution, let us replace $C$ by $C^* = C + \theta S$, where $\theta$ is a non-negative scalar parameter and $S$ is a specified, albeit arbitrary, $(n \times 1)$ vector which determines a given direction of change in the coefficients $c_j, j=1, \ldots, n$. In this regard, the $c_j^*, j=1, \ldots, n$, are specified as linear functions of the parameter $\theta$. Currently $-C'_R + C'_B B^{-1} R \geq 0'$ or $-\bar{c}_j = -c_{Rj} + C'_B Y_j \geq 0, j=1, \ldots, n-m$. If $C^*$ is partitioned as

$$
\begin{bmatrix}
C^*_B \\
C^*_R
\end{bmatrix} = \begin{bmatrix}
C_B \\
C_R
\end{bmatrix} + \theta \begin{bmatrix}
S_B \\
S_R
\end{bmatrix},
$$

where $S_B(S_R)$ contains the components of $S$ corresponding to the components of $C$ within $C_B(C_R)$, then when $C^*$ replaces $C$, the revised optimality condition becomes

$$
-(C'_R)^' + (C'_B)^B^{-1} R = -(C'_R + \theta S'_R) + (C'_B + \theta S'_B)B^{-1} R
\equiv 0'
$$

(12.1)

or, in terms of the individual components of (12.1),

$$
-\bar{c}_j^* = -\bar{c}_j + \theta (-s_{Rj} + S'_B Y_j) \geq 0, j=1, \ldots, n-m.
$$

(Note that the parametrization of $f$ affects only primal optimality and not primal feasibility since $X_B$ is independent of $C$). Let us now determine the largest value of $\theta$ (known as its critical value, $\theta_c$) for which (12.1.1) holds. Upon examining this expression it is evident that the critical value of $\theta$ is that for which any increase in $\theta$ beyond $\theta_c$ makes at least one of the $-\bar{c}_j^*$ values negative, thus violating optimality.

How large of an increase in $\theta$ preserves optimality? First, if $-S'_R + S'_B B^{-1} R \geq 0'$ or $-s_{Rj} + S'_B Y_j \geq 0$, then $\theta$ can be increased without bound while still maintaining the revised optimality criterion since, in this instance, (12.1.1) reveals that $-\bar{c}_j^* \geq -\bar{c}_j \geq 0, j=1, \ldots, n-m$. Next, if $-s_{Rj} + S'_B Y_j < 0$ for some particular value of $j$, then $-\bar{c}_j^* \not\geq 0$ for

$$
\theta \leq \frac{\bar{c}_j}{-s_{Rj} + S'_B Y_j} = \theta_c.
$$